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Unification in Description Logic $\mathcal{EL}$ without top constructor

by

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Declaration
Hereby I certify that the thesis has been written by me. Any help that I have received in my research work has been acknowledged. Additionally, I certify that I have not used any auxiliary sources and literature except those I cited in the thesis.

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Abstract

In recent years, the description logic $\mathcal{EL}$ has received a significant interest. The description logic $\mathcal{EL}$ is a knowledge representation formalism used e.g in natural language processing, configuration of technical systems, databases and biomedical ontologies. Unification is used there as a tool to recognize equivalent concepts. It has been proven that unification in $\mathcal{EL}$ is NP-complete. This result was based on a locality property of certain $\mathcal{EL}$ unifiers. Here we show that a similar locality holds for $\mathcal{EL}$ without top ($\mathcal{EL}$-top), but decidability of $\mathcal{EL}$-top unification does not follow immediately from locality as it does in the case of unification in $\mathcal{EL}$. However, by restricting further the locality property, we prove that $\mathcal{EL}$-top unification is decidable and construct an NExpTime decision procedure for the problem. Furthermore, we show that unification in $\mathcal{EL}$-top is PSPACE-hard by reducing the Finite State Automata Intersection problem to $\mathcal{EL}$-top unification problem.

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Chapter 1

Introduction

1.1 Description Logics

Description Logics (DLs) [2] are a family of logic-based knowledge representation formalisms which can be used to represent concept definitions in a structured and formally well-understood way. The basic notion used to express knowledge in DLs is *concept terms*. Concept terms are built upon *concept names*, *role names* and *concept constructors*. The semantics of concept terms is given by means of an interpretation which consists of a non-empty set of individuals and an interpretation function. In this interpretation, concept names represent sets of individuals and role names represent binary relations between individuals. For example, in order to represent the concept of *women having daughters*, we use **Human**, **Female** as concept names, **hasChild** as a role name and an existential restriction ($\exists$**r.C**) as a concept constructor. The concept term is expressed as follows:

$$Human \sqcap Female \sqcap \exists hasChild.Female$$

By using *value restriction* $\forall r.C$, we can describe the concept of *women having only daughters* :

$$Human \sqcap Female \sqcap \forall hasChild.Female$$

One of the main reasoning problems in DLs is *deciding subsumption*. We say that C is subsumed by D ($C \sqsubseteq D$) if the first concept is always interpreted as a subset of the second one. For example, the following subsumption holds:

$$Human \sqcap Female \sqcap \exists hasChild.Female \sqsubseteq Human \sqcap Female$$

since *all women having daughters are women*. By checking subsumption, we can build a hierarchy graph for concept terms. Moreover, one important application of deciding subsumption is determining *equivalence* between two concept terms. Two concept terms C and D are equivalent if they subsume each other. The equivalence test can be used to detect redundancies in knowledge bases.
Let us consider two following concept terms:

$$Human \sqcap \forall hasChild.Smart \sqcap \forall hasChild.Strong$$
$$Human \sqcap \forall hasChild.(Smart \sqcap Strong)$$

These two concept terms are equivalent, since they both describe the concept of *people having smart and strong children*. However, equivalence testing cannot always determine equivalence of two concept terms, since concepts expressing the same notion could have either different names or different representation. For example, the concept of *men loving adventure sports* could be represented by either the concept term:

$$Human \sqcap Male \sqcap \exists loves.AdventureSports$$

or another one:

$$Man \sqcap \exists loves.(Sport \sqcap Dangerous)$$

These two concept terms are not equivalent. Nevertheless, in order to make them equivalent, we can substitute *AdventureSports* by *Sport* $\sqcap$ *Dangerous* and *Man* by *Human* $\sqcap$ *Male*. In other words, we solve *unification of concept terms* by finding an appropriate substitution (or *unifier*) to make two concept terms equivalent.

1.2 Unification in Description Logics

Unification problems in $\mathcal{EL}$ and their types were originally not introduced for Description Logics, but for equational theories [6]. Unification in DLs was first studied for the description logic $\mathcal{FL}_0$ [5]. $\mathcal{FL}_0$ is a light-weight DL which allows *conjunction* ($\sqcap$), *value restrictions* ($\forall r.C$) and *top concept* ($\top$) as constructors. Unification in $\mathcal{FL}_0$ corresponds to unification modulo the equational theory of idempotent Abelian monoids with several homomorphisms. In [1], it was shown that even for a single homomorphism, unification modulo this theory has *type zero*, i.e., there are unification problems for this theory that do not have a minimal complete set of unifiers. Unification in $\mathcal{FL}_0$ was shown in [5] to be decidable and ExpTime-complete.

Unification in the description logic $\mathcal{EL}$, which uses *existential restrictions* ($\exists r.C$) instead of *value restrictions* ($\forall r.C$), was shown in [5] to be decidable and NP-complete. However, it was also shown there that $\mathcal{EL}$ has unification type zero. Furthermmore, it was shown that the equivalence problem for $\mathcal{EL}$-concept terms corresponds to the word problem for the equational theory of semilattices with monotone operators [14]. From this result, unification in semilattices with monotone operators was shown to be NP-complete and of unification type zero [4].

Even though $\mathcal{EL}$ is inexpressive, it has been used in a variety of applications, e.g., to define biomedical ontologies. In particular, $\mathcal{EL}$ was used to build the large medical ontology SNOMED CT [13] and the Gene Ontology [7] as well as large parts of the medical ontology GALEN [12]. Furthermore, an extension of $\mathcal{EL}$ was used in OWL2 EL which is a sub-profile contained in OWL2 standard [11].

Before unification in $\mathcal{EL}$ was investigated, $\mathcal{EL}$-matching (solving equations whose one side does not contain variables) problem had been considered in [3, 9]. It was shown that deciding whether a given $\mathcal{EL}$-matching problem is solvable or not is NP-complete. Surprisingly, as mentioned above, unification in $\mathcal{EL}$ was shown to be in the same complexity class even though $\mathcal{EL}$-unification is more general than $\mathcal{EL}$-matching.

1.3 Motivation and structure of the thesis

As mentioned above, it was shown that unification in $\mathcal{EL}$ is decidable and NP-complete. The new DL $\mathcal{EL}$-top obtained from $\mathcal{EL}$ by removing top concept from the set of contructors is less expressive than $\mathcal{EL}$. Interestingly, unification in $\mathcal{EL}$-top could be more difficult to solve than that in $\mathcal{EL}$. Let us look at the following unification problem:

$$\Gamma = \{\exists r.X \sqcap X \equiv \exists r.X\}.$$

It is not difficult to see that Γ has a unifier γ in $\mathcal{EL}$, where $\gamma(X) = \top$. But it does not have unifiers in $\mathcal{EL}$-top. We can also say that $\mathcal{EL}$-top unifiers are *stronger* than $\mathcal{EL}$-unifiers in the sense that if an $\mathcal{EL}$-top unification problem has a solution, then it also has a solution in $\mathcal{EL}$. Nevertheless, the other way around does not always hold. Because of this reason, it is worthy to investigate unification in $\mathcal{EL}$-top. In this report, our aim is to solve decidability of $\mathcal{EL}$-top unification problem. To restrict the set of all possible unifiers, we introduce a notion of *local unifiers*. Locality is a property of unifiers which guarantees that an assignment of a local unifier to each variable can be constructed from elements present in the problem and a free concept constant. The similar notion of locality in different contexts was already used for *inference rules sets* [10] and *sets of Horn clauses* [14]. It was shown in [14] that polynomial complexity of deciding subsumption in $\mathcal{EL}$ follows from locality of algebraic models of $\mathcal{EL}$.
The most important goal of this thesis is to show that unification in $\mathcal{EL}$ is decidable and PSPACE-hard. For this purpose, the thesis is organised as follows:

- In Chapter 2, we define formally the description logic $\mathcal{EL}$-top and introduce unification in $\mathcal{EL}$-top,
- In Chapter 3, we present locality of $\mathcal{EL}$-top unification and at the end show the reason why decidability does not follow immediately from the locality.
- In Chapter 4, we show that $\mathcal{EL}$-top unification is decidable and describe a NExpTime decision procedure of $\mathcal{EL}$-top unification problem.
- In Chapter 5, we prove that unification in $\mathcal{EL}$-top is PSPACE-hard by reducing the Finite State Automata Intersection problem to $\mathcal{EL}$-top unification problem.

- In Chapter 6, we summarize the results of this thesis and propose some future work.

Chapter 2

Unification in the description logic $\mathcal{EL}$-top

In this chapter, we define formally the description logic $\mathcal{EL}$-top, compare it to the logic $\mathcal{EL}$ and show that some properties of unifiers in $\mathcal{EL}$ hold also for unifiers in $\mathcal{EL}$-top. Our aim is to define unification problem in $\mathcal{EL}$-top. In the subsequent chapters, we want to prove the decidability of this problem. Hence in this chapter, we present a short description of decidability procedure in $\mathcal{EL}$, explain why it does not work in $\mathcal{EL}$-top and what has to be done to modify it so that such a procedure does solve unification in $\mathcal{EL}$-top.

2.1 Description logic $\mathcal{EL}$-top

We obtain $\mathcal{EL}$-top from $\mathcal{EL}$ by removing $\top$ from the set of constructors and concept terms. In the description logic $\mathcal{EL}$-top, we define concept terms in the following way.

2.1.1 Concept terms

Syntax

Let N_{con} be a set of concept names and N_{role} a set of role names. $\mathcal{EL}$-top concept terms are defined as follows:

- C is an $\mathcal{EL}$-top concept term, for all $C \in N_{con}$.
- If C, D are $\mathcal{EL}$-top concept terms, then so is $C \sqcap D$.
- If C is an $\mathcal{EL}$-top concept term and $r \in N_{role}$, then $\exists r.C$ is an $\mathcal{EL}$-top concept term.

We write $\exists r_1 ... r_k.C$ as an abbreviation of $\exists r_1.\exists r_2 ... \exists r_k.C$, for $k \geq 2$.

Semantics

The semantics of $\mathcal{EL}$ is given by the notion of an interpretation $\mathcal{I} = \{\mathcal{D}_\mathcal{I}, \cdot^\mathcal{I}\}$ consisting of a nonempty domain $\mathcal{D}_\mathcal{I}$ and an interpretation function $\cdot^\mathcal{I}$ which assigns each concept term a subset of $\mathcal{D}_\mathcal{I}$ and each role name a subset of $\mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I}$.

- $A^\mathcal{I} \subseteq \mathcal{D}_\mathcal{I}$, for all $A \in N_{con}$,
- $r^\mathcal{I} \subseteq \mathcal{D}_\mathcal{I} \times \mathcal{D}_\mathcal{I}$, for all $r \in N_{role}$,
- $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$,
- $(\exists r.C)^\mathcal{I} = \{x \mid \exists y : (x, y) \in r^\mathcal{I} \wedge y \in C^\mathcal{I}\}$,

Since conjunction $\sqcap$ is obviously associative and commutative, we omit parenthesises and treat $\sqcap$ as a constructor with flexible arity. In particular, this means that we treat arguments of a conjunction as a multiset. Notice that then we can have a conjunction with one argument only, but in $\mathcal{EL}$-top we cannot have an empty conjunction.
In $\mathcal{EL}$-top, all concepts are satisfiable. The interesting problems are those concerning subsumption and equivalence between concepts.

2.1.2 Subsumption and equivalence

Syntactically, we use the following formulas to express subsumption and equivalence of concepts:

- $C \sqsubseteq D$ (C is subsumed by D), where C, D are $\mathcal{EL}$-concept terms.
- $C \equiv D$ (C is equivalent to D), where C, D are $\mathcal{EL}$-concept terms.
- $C \sqsubset D$ (C is strictly subsumed by D), where C, D are $\mathcal{EL}$-concept terms, iff $C \sqsubseteq D$ and $C \not\equiv D$.

The semantics is given as follows:

- $C \sqsubseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ for all interpretations $\mathcal{I}$,
- $C \equiv D$ iff $C^\mathcal{I} = D^\mathcal{I}$ for all interpretations $\mathcal{I}$.

Syntax and semantics of $\mathcal{EL}$-top are summed up in Table 1.1.

The differences between $\mathcal{EL}$ and $\mathcal{EL}$-top w.r.t. syntax are illustrated in Table 1.2.

Table 2.1: Syntax and semantics of $\mathcal{EL}$-top

Name	Syntax	Semantics
concept name	A	$A^{\mathcal{I}} \subseteq \mathcal{D}_{\mathcal{I}}$
role name	r	$r^{\mathcal{I}} \subseteq \mathcal{D}_{\mathcal{I}} \times \mathcal{D}_{\mathcal{I}}$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists r.C$	$(\exists r.C)^{\mathcal{I}} = \{x \mid \exists y : (x, y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
subsumption	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
equivalence	$C \equiv D$	$C^{\mathcal{I}} = D^{\mathcal{I}}$

Table 2.2: Comparison of $\mathcal{EL}$ and $\mathcal{EL}$-top

Name	$\mathcal{EL}$	$\mathcal{EL}$-top
$\top$	✓	×
$\sqcap$	✓	✓
$\exists$	✓	✓

2.1.3 Atoms

An $\mathcal{EL}$-top concept term is called an **atom** iff it is either a concept name or an existential restriction. It is easy to see that every $\mathcal{EL}$-top concept term is a conjunction of atoms.
Let $At(C)$ be the set of atoms of an $\mathcal{EL}$-top concept term C, then:

- If C is a concept name, then $At(C) := \{C\}$,
- If $C = \exists r.D$, then $At(C) := \{C\} \cup At(D)$,
- If $C = C_1 \sqcap C_2$, then $At(C) := At(C_1) \cup At(C_2)$.

An atom is called **flat atom** if it is either a concept name or an existential restriction $\exists r.D$, where D is a concept name.

2.1.4 Reduction rules

The following reduction rules preserve equivalence of concepts in $\mathcal{EL}$-top modulo associativity and commutativity of $\sqcap$:

- $A \sqcap A \rightarrow A$, for concept names $A \in N_{con}$,
- $\exists r.C \sqcap \exists r.D \rightarrow \exists r.C$, for all $\mathcal{EL}$-top concept terms C, D with $C \sqsubseteq D$.

An $\mathcal{EL}$-top concept term D is called **reduced** if it is obtained by an exhaustive application of the above rules. D is a **reduced form** of an $\mathcal{EL}$-top concept term C if D is obtained from C by applying the above rules and D is reduced.
Note that in $\mathcal{EL}$ we have an additional reduction rule: $A \sqcap \top \rightarrow A$. Since we do not

have $\top$ in $\mathcal{EL}$-top, the additional rule is never applicable and hence the following theorem known for $\mathcal{EL}$ can be easily obtained as a consequence of Theorem 6.3.1 in [9].

Theorem 1. *Let C, D be $\mathcal{EL}$-top concept terms, and $\hat{C}, \hat{D}$ reduced forms of C, D, respectively. Then $C \equiv D$ iff $\hat{C}$ is identical to $\hat{D}$ up to associativity and commutativity of $\sqcap$.*

The following result is very useful in proving properties associated to subsumption in $\mathcal{EL}$-top.

Corollary 1. *Let $C = A_1 \sqcap ... \sqcap A_k \sqcap \exists r_1.C_1 \sqcap .... \sqcap \exists r_m.C_m$ and $D = B_1 \sqcap ... \sqcap B_l \sqcap \exists s_1.D_1 \sqcap .... \sqcap \exists s_n.D_n$, where $A_1, ..., A_k, B_1, .., B_l$ are concept names. Then $C \sqsubseteq D$ iff $\{B_1, ..., B_l\} \subseteq \{A_1, ..., A_k\}$ and for every $j, 1 \leq j \leq n$, there exists an $i, 1 \leq i \leq m$, such that $r_i = s_j$ and $C_i \sqsubseteq D_j$.*

2.2 Unification in $\mathcal{EL}$-top

The notions of variables and substitutions for $\mathcal{EL}$-top are similar to those for $\mathcal{EL}$ [4].
We assume that the set of concept names is partitioned into two disjoint subsets N_v and N_c, where:

- N_v: A set of concept variables.
- N_c: A set of concept constants.

Definition 1. (Substitutions)
A ***substitution*** *σ is a mapping from N_v into the set of all $\mathcal{EL}$-top concept terms. This mapping is extended to concept terms as follows:*

- *$\sigma(A) := A$, for all $A \in N_c$,*
- *$\sigma(C \sqcap D) := \sigma(C) \sqcap \sigma(D)$,*
- *$\sigma(\exists r.C) := \exists r.\sigma(C)$.*

Definition 2. ($\mathcal{EL}$-top unification problems)
An $\mathcal{EL}$-top unification problem is of the form $\Gamma = \{C_1 \equiv D_1, ..., C_n \equiv D_n\}$, where C_i, D_i are $\mathcal{EL}$-top concept terms, for all $i, 1 \leq i \leq n$. The substitution σ is a ***unifier (solution)*** *of Γ iff $\sigma(C_i) \equiv \sigma(D_i)$, for all $i, 1 \leq i \leq n$. Γ is called* ***solvable*** *or* ***unifiable*** *iff there exists such a σ.*

The following notions are similar to those used in the analysis of $\mathcal{EL}$-unification in [4].

- **Flat $\mathcal{EL}$-top unification problems**
  An $\mathcal{EL}$-top unification problem Γ is **flat** iff it consists of equations between flat $\mathcal{EL}$-top concept terms. Every $\mathcal{EL}$-top unification problem Γ can be transformed in polynomial time into an equivalent flat $\mathcal{EL}$-top unification problem Γ' in the sense that Γ is solvable iff Γ' is solvable. So we can assume that an $\mathcal{EL}$-top unification problem is flat.

- **Reduced (ground) unifiers**
  Let $\Gamma = \{C_1 \equiv^? D_1, ..., C_n \equiv^? D_n\}$ be a flat $\mathcal{EL}$-top unification problem. We call the atoms of $C_1, D_1, ..., C_n, D_n$ the **atoms** of Γ.
  An atom C of Γ is called a **non-variable** of Γ iff C is either a concept constant or an existential restriction. We use the name **non-variable** as an abbreviation for **non-variable atom**.
  The unifier σ of Γ is called **reduced (ground)** iff, for all concept variables X occurring in Γ, the $\mathcal{EL}$-top concept term $\sigma(X)$ is reduced (ground).

- **The atoms of a ground unifier**
  Let σ be a ground unifier of Γ. Then $At(\sigma) = \bigcup At(\sigma(X))$, where X ranges over all variables occurring in Γ, denotes the set of atoms of σ.

- **Minimal ground unifiers**
  We now define the order $>_{is}$ on $\mathcal{EL}$-top concept terms. Let C, D be $\mathcal{EL}$-top concept terms, then $C >_{is} D$ iff $C \sqsubset D$. The strict order $>_{is}$ is well-founded and its multiset extension $>_m$ is also well-founded.
  The following results known for $\mathcal{EL}$ still hold in $\mathcal{EL}$-top.

  Lemma 1. [*4*]
  Let C, D, D' be $\mathcal{EL}$-top concept terms such that D is a reduced atom, $D >_{is} D'$, C is reduced and contains at least one occurrence of D modulo AC. If C' is obtained from C by replacing all occurrences of D by D', then $C >_{is} C'$.

  Let σ be a ground unifier of Γ and $S(\sigma)$ the multiset of all $\mathcal{EL}$-top concept terms $\sigma(X)$, where X ranges over all variables occurring in Γ. We say that $\sigma \succ \theta$ iff $S(\sigma) >_m S(\theta)$, where σ, θ are ground unifiers of Γ.
  A ground unifier σ of Γ is called **minimal** iff there is no ground unifier θ of Γ such that $\sigma \succ \theta$. The following proposition shows that the decidability of an $\mathcal{EL}$-top unification problem can be reduced to one w.r.t. minimal reduced ground unifiers.

  Proposition 1. *Let Γ be an $\mathcal{EL}$-top unification problem. Then Γ is solvable iff it has a minimal reduced ground unifier.*

2.3 Unification in $\mathcal{EL}$: previous results

Unification in $\mathcal{EL}$ is local in the sense that each minimal reduced ground unifier of an $\mathcal{EL}$-unification problem is constructed from elements present in the problem.

This follows from the following lemma.

Lemma 2. *Let Γ be a flat $\mathcal{EL}$-unification problem and γ a minimal reduced ground unifier of Γ. If X is a variable occurring in Γ, then $\gamma(X) = \top$ or there are non-variables $D_1, ..., D_n$ of Γ such that $\gamma(X) = \gamma(D_1) \sqcap ... \sqcap \gamma(D_n)$.*

Because of this lemma, we can define an NP-algorithm that decides the unification problem in $\mathcal{EL}$ and computes all minimal reduced ground unifiers.
Given a flat $\mathcal{EL}$-unification problem Γ, a substitution σ can be computed by the algorithm as follows:

1. For each variable X occurring in Γ, guess a finite set S_X of non-variables of Γ,

2. We define **depends on** relation over the variables occurring in Γ: A variable X **directly depends on** a variable Y if Y occurs in an element of S_X. Let *depends on* be the transitive closure of *directly depends on*. If there is a variable that depends on itself, then the algorithm returns "fail". Otherwise, we define a strict linear order $>$ on the variables occurring in Γ such that $X > Y$ if X depends on Y.

3. By the assumption that $>$ is strict, σ is computed as follows:

   - If X is a minimal variabe w.r.t. $>$. Then $\sigma(X) = \bigsqcap_{D \in S_X} D$ if $S_X \neq \emptyset$. Otherwise, $\sigma(X) = \top$.
   - For all $Y < X$, we assume that $\sigma(Y)$ is already defined. Then $\sigma(X) = \bigsqcap_{D \in S_X} \sigma(D)$ if $S_X \neq \emptyset$. Otherwise, $\sigma(X) = \top$.

4. Test whether σ is a unifier of Γ or not. If σ is a unifier, then return σ. Otherwise, return "fail".

The algorithm is sound and complete [4]. Moreover, it always terminates. Thus it is a decision procedure for $\mathcal{EL}$-unification. For the case of $\mathcal{EL}$-top unification, this algorithm does not work in general, because the substitution σ may assign $\top$ to variables.
To obtain a similar algorithm for $\mathcal{EL}$-top unification, we have to introduce some **minimal concept terms** as a replacement for $\top$. Such concept terms are defined formally in the next chapter.

Chapter 3

Locality of unification in $\mathcal{EL}$-top

In this chapter, the main theorem is Theorem 2. It says that if a given flat $\mathcal{EL}$-top unification problem is solvable, then it has a minimal **local unifier** which is defined formally later. Intuitively, we understand **local unifier** as one which can be constructed from elements in the goal. In order to prove it, we have to show the following:

- Each minimal reduced ground unifier is almost local, i.e., it may contains some ”non-local” atoms.
- By a replacement of those atoms, we always obtain a minimal local unifier.

At the end of this chapter, we show that locality is not strong enough to obtain a decision procedure for $\mathcal{EL}$-top unification.

3.1 Notation

In the analysis of $\mathcal{EL}$-top unification, we use the following notions.

Definition 3. (Set of constants $Cons$)
Let Γ be an $\mathcal{EL}$-top unification problem. A set of constants ***Cons(*Γ*)*** *is defined as follows:*

- *If there exists a concept constant in Γ, then $Cons(\Gamma) = \{A \mid A \text{ is in } \Gamma\}$.*
- *Otherwise, $Cons(\Gamma) = \{A\}$, where A is a new concept constant.*

Now we introduce the notion of **comparable atoms**.

Definition 4. (Comparable atoms)
Let Γ be an $\mathcal{EL}$-top unification problem, γ a reduced ground unifier of Γ and C an atom of γ. We say that C is ***comparable*** *to Γ iff there is a non-variable D of Γ such that either $C \sqsubseteq \gamma(D)$ or $\gamma(D) \sqsubseteq C$. Otherwise, C is called* ***incomparable*** *to Γ.*

3.2 Subatoms and their properties

In $\mathcal{EL}$, $\top$ is the maximal concept term w.r.t. $\sqsubseteq$. We now define a set of subatoms which are minimal elements w.r.t. inverse of subsumption order in $\mathcal{EL}$-top.

Definition 5. (Subatoms)
Let C be an $\mathcal{EL}$-top concept term. We say that C is a ***subatom*** *iff:*

- *C is a concept name, or*
- *$C = \exists r.C'$, where C' is a subatom.*

Definition 6. (Set of subatoms of a concept term)
Let T be the set of all $\mathcal{EL}$-top concept terms. For $C, C_1, C_2 \in T$, we define a function $SubAt : T \to 2^T$ as follows :

- *$SubAt(C) = \{C\}$, where C is a concept name.*
- *$SubAt(\exists r.C) = \{\exists r.M \mid M \in SubAt(C)\}$.*
- *$SubAt(C_1 \sqcap C_2) = SubAt(C_1) \cup SubAt(C_2)$.*

We say that $D \in SubAt(C)$ is a ***subatom*** *of C.*

A subatom cannot be strictly subsumed by any concept terms. It is shown by the following lemma.

Lemma 3. *Let C is a subatom and B a concept term. Then $C \sqsubseteq B$ implies that $C \equiv B$.*

Proof. We prove the lemma by induction on the structure of C. We assume that $C \sqsubseteq B$.

- If C is a concept constant, then it is obvious that $C \sqsubseteq B$ implies that $C \equiv B$.
- If $C = \exists r.C'$ for some subatom C', then by Corollary 1, $C \sqsubseteq B$ implies that there is a concept term B' such that $B = \exists r.B'$ and $C' \sqsubseteq B'$. Since C' is a subatom, by induction we have $C' \sqsubseteq B'$ implies that $C' \equiv B'$. Thus $C = \exists r.C' \equiv \exists r.B' = B$.

□

The following lemma shows that subsumption of atoms implies an inclusion of the sets of their subatoms.

Lemma 4. *Let A, B be atoms. Then $A \sqsubseteq B$ implies that $SubAt(B) \subseteq SubAt(A)$.*

Proof. We prove the lemma by induction on the structure of A.

- A is a concept constant. Then $A \sqsubseteq B$ implies that $A = B$ and thus we have $SubAt(B) = SubAt(A)$ which implies that $SubAt(B) \subseteq SubAt(A)$.

- $A = \exists r.A'$ for some concept term A'. Since B is an atom, $A \sqsubseteq B$ implies that there exists a concept term B' such that $B = \exists r.B'$ and $A' \sqsubseteq B'$. Let:
  - $A' = A_1 \sqcap ... \sqcap A_n$,
  - $B' = B_1 \sqcap ... \sqcap B_m$,

  where $A_1, ..., A_n, B_1, ..., B_m$ are atoms.

  (i) Since $A' \sqsubseteq B'$, for each j, $1 \leq j \leq m$, there is an i, $1 \leq i \leq n$ such that $A_i \sqsubseteq B_j$. By induction, we have $SubAt(B_j) \subseteq SubAt(A_i)$.

  (ii) On the other hand, by definition of SubAt (Definition 6), we have:
  - $SubAt(B) = \{\exists r.D \mid D \in \bigcup_{1 \leq j \leq m} SubAt(B_j)\}$ and
  - $SubAt(A) = \{\exists r.D \mid D \in \bigcup_{1 \leq i \leq n} SubAt(A_i)\}$.

  By (i) and (ii), we have $SubAt(B) \subseteq SubAt(A)$.

□

The next lemma explains the relation between a concept term and its subatoms. Moreover, the second claim is indeed a generalization of Lemma 4.

Lemma 5. *Let T be an $\mathcal{EL}$-top concept term. Then the following holds:*

1. *$T \sqsubseteq C$, for all $C \in SubAt(T)$.*
2. *If $T \sqsubseteq D$ for some $\mathcal{EL}$-top concept term D, then $SubAt(D) \subseteq SubAt(T)$.*

Proof. We first prove (1). From the definition, it is easy to see that for any $\mathcal{EL}$-top concept term T we have $SubAt(T) \neq \emptyset$.
Let $C \in SubAt(T)$. We prove (1) by induction on the structure of T.

- If $T = A$, for a concept name A, then $SubAt(T) = \{A\}$ and $T = A = C$ implies that $T \sqsubseteq C$.
- If $T = \exists r.A$, for some $\mathcal{EL}$-top concept term A, then by induction, we have $A \sqsubseteq B$ for all $B \in SubAt(A)$. On the other hand, we have $SubAt(T) = \{\exists r.B \mid B \in SubAt(A)\}$. Thus, $\exists r.A \sqsubseteq C$.
- Assume that $T = T_1 \sqcap T_2$, with $\mathcal{EL}$-top concept terms T_1, T_2. Since $SubAt(T) = SubAt(T_1) \cup SubAt(T_2)$, $C \in SubAt(T)$ implies that there exists a k, $1 \leq k \leq 2$ such that $C \in SubAt(T_k)$. By induction, we have $T_k \sqsubseteq C$. Thus, $T = T_1 \sqcap T_2 \sqsubseteq T_k \sqsubseteq C$.

We now prove (2). Assume that:

- $T = T_1 \sqcap ... \sqcap T_n$,
- $D = D_1 \sqcap ... \sqcap D_m$,

where $T_1, ..., T_n, D_1, ..., D_m$ are atoms.

(i) Since $T \sqsubseteq D$, for each j, $1 \leq j \leq m$, there is an i, $1 \leq i \leq n$ such that $T_i \sqsubseteq D_j$. By Lemma 4, we have $SubAt(D_j) \subseteq SubAt(T_i)$.

(ii) On the other hand, by definition of SubAt (Definition 6), we have:

- $SubAt(D) = \{\exists r.C \mid C \in \bigcup_{1 \leq j \leq m} SubAt(D_j)\}$ and
- $SubAt(T) = \{\exists r.C \mid C \in \bigcup_{1 \leq i \leq n} SubAt(T_i)\}$.

By (i) and (ii), we have $SubAt(D) \subseteq SubAt(T)$ which completes the proof of the lemma. □

3.3 Operations on concept terms

An operation o on concept terms is a function which assigns a concept term to a concept term. In order to obtain unifiers with desired properties, we introduce two operations on concept terms:

- **Replacement** : Replace every occurrence of an atom in a concept term by some concept term. In particular, by $E^{[C/D]}$, we denote the concept term obtained from a concept term E by replacing all occurrences of C by D.

- **Deletion** : Remove every atom, which satisfies certain condition, from a concept term.

3.3.1 Replacement

The following lemma says that with certain condition, replacing one subatom by another in a concept term preserves subsumption.

Lemma 6. *Let A be a concept term, C, C', B subatoms such that C' and B does not contain C. Then $A \sqsubseteq B$ implies that $A^{[C/C']} \sqsubseteq B$.*

Proof. If $A = C$ then since C is a subatom, $A \sqsubseteq B$ implies that $A = B$ and thus $B = C$ which contradicts our assumption on B. Therefore, $A \neq C$. We now prove the lemma by induction on the structure of A.

- A is a concept constant. Then $A \sqsubseteq B$ implies that $A = B$ and thus $A^{[C/C']} = B^{[C/C']} = B$ implies that $A^{[C/C']} \sqsubseteq B$.

- $A = \exists r.A'$ for some concept term A'. Since $A \neq C$, $A^{[C/C']} = \exists r.A'^{[C/C']}$. Since $A \sqsubseteq B$ and B is a subatom, there is a subatom B' such that $A' \sqsubseteq B'$ and $B = \exists r.B'$. Since B does not contain C, neither does B'. By induction, we have $A'^{[C/C']} \sqsubseteq B'$. Thus $A^{[C/C']} = \exists r.A'^{[C/C']} \sqsubseteq \exists r.B' = B$.

- $A = A_1 \sqcap A_2$ for some concept terms A_1, A_2. Since B is an atom, $A \sqsubseteq B$ implies that either $A_1 \sqsubseteq B$ or $A_2 \sqsubseteq B$. Since $A \neq C$, $A^{[C/C']} = A_1^{[C/C']} \sqcap A_2^{[C/C']}$. Without loss of generality, we assume that $A_1 \sqsubseteq B$. By induction, $A_1^{[C/C']} \sqsubseteq B$ and thus $A^{[C/C']} = A_1^{[C/C']} \sqcap A_2^{[C/C']} \sqsubseteq B$.

□

3.3.2 Deletion

Definition 7. (C-deletion)
Let C be a subatom and D a concept term. We define a ***C-deletion*** D^{-C} *of a concept term D as follows:*

- D^{-C} *is undefined if* $D = C$,
- *if* $D \neq C$, *then*
  - *if D is a constant, then* $D^{-C} = D$.
  - $(\exists r.D)^{-C} = \exists r.D^{-C}$ *if* D^{-C} *is defined. Otherwise,* $(\exists r.D)^{-C}$ *is undefined.*
  - $(D_1 \sqcap D_2)^{-C} = D_1^{-C} \sqcap D_2^{-C}$ *if* D_1^{-C} *and* D_2^{-C} *are defined,*
  - $(D_1 \sqcap D_2)^{-C} = D_1^{-C}$ *if* D_1^{-C} *is defined and* D_2^{-C} *is undefined,*
  - $(D_1 \sqcap D_2)^{-C} = D_2^{-C}$ *if* D_1^{-C} *is undefined and* D_2^{-C} *is defined.*

We say that D is C-defined w.r.t. C iff D^{-C} is defined.
The following result shows that C-deletion preserves subsumption.

Lemma 7. *Let C be a subatom and* A, B *atoms. If* $A \sqsubseteq B$ *and* B^{-C} *is defined then* A^{-C} *is defined and* $A^{-C} \sqsubseteq B^{-C}$.

Proof. We prove the lemma by induction on the structure of B.

- If B is a constant, then since A is an atom, $A \sqsubseteq B$ implies that $A = B$. Thus B^{-C} is defined implies that A^{-C} is defined. Moreover, we have $A^{-C} = B^{-C}$ which yields that $A^{-C} \sqsubseteq B^{-C}$.

- If $B = \exists r.B_1 \sqcap ... \sqcap B_n$, where $B_1, ..., B_n$ are atoms, then since A is an atom, $A \sqsubseteq B$ implies that $A = \exists r.A_1 \sqcap ... \sqcap A_m$, where $A_1, ..., A_m$ are atoms. On the other hand, for each B_i, there is A_j such that $A_j \sqsubseteq B_i$. Thus if B_i^{-C} is defined, then by induction we have A_j^{-C} is defined and $A_j^{-C} \sqsubseteq B_i^{-C}$. Since B^{-C} is defined, there exists an i, $1 \leq i \leq m$ such that B_i^{-C} is defined and thus A_j^{-C} is defined. It yields that A^{-C} is defined and $A^{-C} \sqsubseteq B^{-C}$.

□

Intuitively, by applying C-deletion, we can obtain a *bigger* concept term w.r.t. $\sqsubseteq$. More precisely, we prove the following lemma.

Lemma 8. *Let A be a concept term and C a subatom occurring in A. Then A^{-C} is defined implies that $A \sqsubset A^{-C}$.*

Proof. We asssume that A^{-C} is defined. Thus $A \neq C$. It is easy to see that A can not be a concept constant. We prove the lemma by induction on the structure of A.

- $A = \exists r.A'$ for some concept term A'. Since $A \neq C$, we have $A^{-C} = \exists r.A'^{-C}$. Since A^{-C} is defined, A'^{-C} is defined. Moreover, since $A \neq C$, C occurs in A'. By induction we have $A' \sqsubset A'^{-C}$ and thus $A \sqsubset \exists r.A'^{-C} = A^{-C}$.

- $A = A_1 \sqcap A_2$ for some concept terms A_1, A_2. Since $A \neq C$ and A^{-C} is defined, either A_1^{-C} or A_2^{-C} is defined. Since C is an atom, C occurs in either A_1 or A_2. Without loss of generality, we assume that C occurs in A_1.

  - If A_1^{-C} is defined, then by induction we have $A_1 \sqsubset A_1^{-C}$. On the other hand, if C occurs in A_2 and A_2^{-C} is defined, then by induction we have $A_2 \sqsubset A_2^{-C}$ and thus $A = A_1 \sqcap A_2 \sqsubset A_1^{-C} \sqcap A_2^{-C} = A^{-C}$. Otherwise, if A_2^{-C} is not defined, then $A^{-C} = A_1^{-C}$ and thus $A = A_1 \sqcap A_2 \sqsubset A_1^{-C} = A^{-C}$. Moreover, if C does not occur in A_2, then $A_2^{-C} = A_2$ and thus $A = A_1 \sqcap A_2 \sqsubset A_1^{-C} \sqcap A_2 = A^{-C}$.
  - If A_1^{-C} is not defined, then A_2^{-C} is defined and $A^{-C} = A_2^{-C}$. If C occurs in A_2, then by induction, we have $A_2 \sqsubset A_2^{-C}$ and thus $A = A_1 \sqcap A_2 \sqsubset A_2 \sqsubset A_2^{-C} = A^{-C}$. Otherwise, $A_2^{-C} = A_2$ and thus $A = A_1 \sqcap A_2 \sqsubset A_2 = A^{-C}$.

□

The following lemma can be shown in a similar way as for Lemma 8, but we do not require C to occur in A.

Lemma 9. *Let C be a subatom. Then for every concept term A, A^{-C} is defined implies that $A \sqsubseteq A^{-C}$.*

3.4 Operations on substitutions

We extend an operation on concept terms to substitutions as follows.

Definition 8. (Operation on substitutions)
Let Γ be an $\mathcal{EL}$-top unification prolem, γ a reduced ground unifier of Γ and o an operation on concept terms. We denote γ^o the substitution obtained from γ by applying o in such a way that $\gamma^o(X) = (\gamma(X))^o$, where C^o denotes the concept term obtained from a concept term C by applying o to C.

3.4.1 Replacement

The following lemma shows that under the necessary conditions, the substitution obtained from some reduced ground unifier by applying replacement to the unifier is also a unifier.

Lemma 10. *Let Γ be a flat $\mathcal{EL}$-top unification prolem, γ a reduced ground unifier of Γ. Let C be an atom of γ such that there is no non-variable D of Γ with $\gamma(D) \equiv C$ and o any replacement of C on concept terms. If $A \sqsubseteq B$ implies that $A^o \sqsubseteq B^o$, where each A, B is either an atom of γ or equal to $\gamma(D)$ for some non-variable D of Γ, then γ^o is a unifier of Γ.*

Proof. We consider an equation in Γ of the form $L_1 \sqcap ... \sqcap L_m \equiv R_1 \sqcap ... \sqcap R_n$, where $L_1, ..., L_m$ and $R_1, ..., R_n$ are flat atoms. We have $L = \gamma(L_1 \sqcap ... \sqcap L_m) = A_1 \sqcap ... \sqcap A_\mu$ and $R = \gamma(R_1 \sqcap ... \sqcap R_n) = B_1 \sqcap ... \sqcap B_\nu$, where each $A_1, ..., A_\mu$ and $B_1, ..., B_\nu$ is a reduced atom that is either an atom of γ or equal to $\gamma(E)$ for a non-variable E of Γ. Since γ is a unifier of Γ, we have $L = R$.
Since C is an atom, we have $L^o := A_1^o \sqcap ... \sqcap A_\mu^o$ and $R^o := B_1^o \sqcap ... \sqcap B_\nu^o$.
By the first condition, for all i, j, $1 \le i \le \mu$, $1 \le j \le \nu$, we have $A_i \sqsubseteq B_j$ implies that

$$A_i^o \sqsubseteq B_j^o. \tag{3.1}$$

We now show that

$$\gamma(K)^o = \gamma^o(K), \text{ for each atom } K \text{ of } \Gamma. \tag{3.2}$$

Since Γ is flat, we have three cases:

1. If $K = A$ for some concept constant A, then $\gamma(K)^o = A = \gamma^o(K)$.
2. If $K = X$ for some variable X, then it is obvious that $\gamma(K)^o = \gamma^o(K)$.
3. Assume that $K = \exists s.M$, M is either a concept constant or a variable. By the assumption on C, we have $\gamma(K) \neq_{AC} C$. That means C can only occur in $\gamma(M)$. By 1 and 2, we have $(\gamma(M))^o = \gamma^o(M)$. Thus, $\gamma(K)^o = \exists s.(\gamma(M))^o = \exists s.(\gamma^o(M)) = \gamma^o(K)$.

By (3.2), we have $\gamma(L_i)^o = \gamma^o(L_i)$, for all $i, 1 \le i \le m$ and $\gamma(R_i)^o = \gamma^o(R_i)$, for all $i, 1 \le i \le n$. Thus we have $L^o = (\gamma(L_1 \sqcap ... \sqcap L_m))^o = (\gamma(L_1) \sqcap ... \sqcap \gamma(L_m))^o = \gamma(L_1)^o \sqcap ... \sqcap \gamma(L_m)^o = \gamma^o(L_1) \sqcap ... \sqcap \gamma^o(L_m) = \gamma^o(L_1 \sqcap ... \sqcap L_m)$.
Similarly, we also have $\gamma^o(R_1 \sqcap ... \sqcap R_n) = R^o$. To show that γ^o is a unifier of Γ, it is enough to prove that $L^o \equiv R^o$.
Without loss of generality, we show only that $L^o \sqsubseteq R^o$. By Corollary 1, it is enough to show that for every j, $1 \le j \le \nu$, there exists an i, $1 \le i \le \mu$ such that $A_i^o \sqsubseteq B_j^o$.
Since $L \sqsubseteq R$, we know that for every j, $1 \le j \le \nu$, there exists an i, $1 \le i \le \mu$ such that $A_i \sqsubseteq B_j$. On the other hand, by (3.1), $A_i \sqsubseteq B_j$ implies that $A_i^o \sqsubseteq B_j^o$. It shows that $L^o \sqsubseteq R^o$.
Thus, γ^o is a unifier of Γ. □

In the next lemma, we show that w.r.t. more specific conditions, assumptions of Lemma 10 are satisfied and a replacement gives us a unifier.

Lemma 11. *Let Γ be a flat $\mathcal{EL}$-top unification problem and γ a reduced ground unifier of Γ. Let C be an atom of γ, $\hat{D}$ an $\mathcal{EL}$-top concept term with $C \sqsubseteq \hat{D}$. We assume that for every non-variable D of Γ and for every atom C' of γ:*

(i) $C \sqsubseteq \gamma(D)$ *implies that* $\hat{D} \sqsubseteq \gamma(D)$*,*

(ii) $C \sqsubset C'$ *implies that* $\hat{D} \sqsubseteq C'$*,*

Then $\gamma^{[C/\hat{D}]}$ *is a unifier of* Γ.

Proof. First, by induction on the size of A, we show that $A \sqsubseteq B$ implies that

$$A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}. \tag{3.3}$$

where each A, B is either an atom of γ or equal to $\gamma(D)$ for some non-variable D of Γ.

1. Assume that $A =_{AC} C$, then $A^{[C/\hat{D}]} = \hat{D}$.

   (a) If $B = \gamma(D)$ for some non-variable D of Γ, then by (i) $C \sqsubseteq \gamma(D)$ implies that $\hat{D} \sqsubseteq B$. Thus, $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

   (b) Assume B is an atom of γ. If $B =_{AC} C$, then $A^{[C/\hat{D}]} = B^{[C/\hat{D}]}$ implies that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$. Otherwise, by (ii) $C \sqsubset B$ implies that $\hat{D} \sqsubseteq B$. It yields that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

   On the other hand, since $C \sqsubseteq \hat{D}$, by Lemma 1, we have $B \sqsubseteq B^{[C/\hat{D}]}$. Thus, $A \sqsubseteq B$ implies that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

2. Now we consider $A \neq_{AC} C$. Since A is an atom, A is either a concept constant or an existential restriction. If A is a concept constant, then by Lemma 3, $A \sqsubseteq B$ implies that $A = B$ and thus $A^{[C/\hat{D}]} = B^{[C/\hat{D}]}$. In particular, $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$. Otherwise, $A = \exists s.E$ and C occurs (modulo AC) in E. Then $A \sqsubseteq B$ yields that $B = \exists s.F$ with $E \sqsubseteq F$. Since E and F are conjunctions of atoms of γ, we have $E = E_1 \sqcap ... \sqcap E_p$ and $F = F_1 \sqcap ... \sqcap F_q$. Then $E \sqsubseteq F$ implies that for each $u, 1 \leq u \leq q$, there exists $v, 1 \leq v \leq p$ such that $E_v \sqsubseteq F_u$. By induction, we assume that $E_v \sqsubseteq F_u$ implies that $E_v^{[C/\hat{D}]} \sqsubseteq F_u^{[C/\hat{D}]}$. Thus, for each $u, 1 \leq u \leq q$, there exists $v, 1 \leq v \leq p$ such that

$$E_v^{[C/\hat{D}]} \sqsubseteq F_u^{[C/\hat{D}]} \tag{3.4}$$

   (a) If $B \neq_{AC} C$, then $A^{[C/\hat{D}]} = \exists s.(E_1^{[C/\hat{D}]} \sqcap ... \sqcap E_p^{[C/\hat{D}]})$ and $B^{[C/\hat{D}]} = \exists s.(F_1^{[C/\hat{D}]} \sqcap ... \sqcap F_q^{[C/\hat{D}]})$ along with (3.4) yield that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

   (b) If $B =_{AC} C$, then C cannot occur (modulo AC) in any of $F_1, ..., F_q$ (otherwise, role depth of B is strictly larger than role depth of C). Thus, we have $B = \exists s.(F_1^{[C/\hat{D}]} \sqcap ... \sqcap F_q^{[C/\hat{D}]})$. Moreover, by (3.4), we obtain

$$A^{[C/\hat{D}]} \sqsubseteq B \tag{3.5}$$

   Since $C \sqsubseteq \hat{D}$, we have

$$B \sqsubseteq B^{[C/\hat{D}]} \tag{3.6}$$

   By (3.5) and (3.6), we have $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

By Lemma 10, $\gamma^{[C/\hat{D}]}$ is a unifier of Γ. □

3.4.2 Deletion

By applying deletion to unifiers, we can obtain C-defined substitutions. Note that unifers obtained by applying deletion are always defined in $\mathcal{EL}$, where an empty conjunction is understood as $\top$. However, in $\mathcal{EL}$-top, we do not allow empty conjunctions assigned to any variables. More formally, C-defined substitutions in $\mathcal{EL}$-top are defined as follows:

Definition 9. (C-defined substitutions)
*Let Γ be a flat $\mathcal{EL}$-top unification problem, C a subatom and γ a reduced ground unifier of Γ. Let γ' be a substitution obtained from γ such that for every variable X occurring in Γ, $\gamma'(X) = \gamma(X)^{-C}$. We say that γ' is **C-defined** w.r.t. (γ, C) iff $\gamma'(X)$ is defined for all X occurring in Γ.*

We now define dependency relation and the dependency order $>$ among variables occurring in a flat $\mathcal{EL}$-top unification problem Γ.

Definition 10. (Dependency relation and the dependency order $>$)
Let Γ be a flat $\mathcal{EL}$-top unification problem and γ a reduced ground unifier of Γ. For every variable X occurring in Γ, we define the set $NV(X)$, where $D \in NV(X)$ iff $\gamma(X) = \gamma(D) \sqcap B$ with a ground concept term B, or $\gamma(X) = \gamma(D)$.
*For two variables X, Y occurring in Γ, we say X **directly depends on** Y iff $\exists r.Y \in NV(X)$ for some role name r. Moreover, X **depends on** Y iff either X directly depends on Y or X directly depends on Z and Z depends on Y for some variable Z.*
If there is no variable that depends on itself then we define the dependency order $>$ among variables occurring in Γ such that $X > Y$ iff X depends on Y.

In fact, for a unifier γ and a subatom C, to check whether a C-defined substitution γ' w.r.t. (γ, C) is defined or not, it is enough to check if γ' is defined for all minimal variables w.r.t. the dependency order $>$.

Lemma 12. *Let Γ be a flat $\mathcal{EL}$-top unification problem, C a subatom and γ a unifier of Γ. Then γ' is C-defined w.r.t. (γ, C) iff $\gamma'(X)$ is defined for every minimal variable X w.r.t. the dependency order $>$.*

Proof. We prove the lemma by showing two directions.

- $\Rightarrow$: Since γ' is C-defined, $\gamma'(X)$ is defined for each variable X occurring in Γ, in particular, for all minimal variables w.r.t. the dependency order $>$.
- $\Leftarrow$: We assume that X is not minimal and $\gamma'(Y)$ is defined for each variable Y such that $X > Y$. Since X is not minimal, there exists at least one variable Z such that $X > Z$ and since $\gamma'(Z)$ is defined, $\gamma'(X)$ is also defined.

□

The following lemma shows a similar result to that for *replacement* on substitutions (Lemma 10) namely that under certain conditions, the substitution obtained from some unifier by applying replacement to the unifier is also a unifier.

Lemma 13. *Let Γ be a flat $\mathcal{EL}$-top unification prolem, γ a reduced ground unifier of Γ and o a C-deletion on concept terms w.r.t. a subatom C. If the following holds:*

1. *$\gamma(L)$ is C-defined w.r.t. C, for each atom L of Γ,*

2. *$\gamma(L)^o = \gamma^o(L)$, for each atom L of Γ,*

then γ^o is a unifier of Γ.

Proof. We consider an equation in Γ of the form $L_1 \sqcap ... \sqcap L_m \equiv R_1 \sqcap ... \sqcap R_n$, where $L_1, ..., L_m$ and $R_1, ..., R_n$ are flat atoms. We have $L = \gamma(L_1 \sqcap ... \sqcap L_m) = A_1 \sqcap ... \sqcap A_\mu$ and $R = \gamma(R_1 \sqcap ... \sqcap R_n) = B_1 \sqcap ... \sqcap B_\nu$, where each $A_1, ..., A_\mu$ and $B_1, ..., B_\nu$ is a reduced atom that is either an atom of γ or equal to $\gamma(E)$ for a non-variable E of Γ. Since γ is a unifier of Γ, we have $L = R$. By condition 1 and 2, for each i, $1 \leq i \leq m$, we have

$$\begin{aligned} &\gamma(L_i)^o \text{ is defined,} \\ &\gamma(L_i)^o = \gamma^o(L_i) \end{aligned} \tag{3.7}$$

Similarly, for each i, $1 \leq i \leq n$, we have

$$\begin{aligned} &\gamma(R_i)^o \text{ is defined,} \\ &\gamma(R_i)^o = \gamma^o(R_i) \end{aligned} \tag{3.8}$$

We define $L^o = (\gamma(L_1 \sqcap ... \sqcap L_m))^o$ and $R^o = (\gamma(R_1 \sqcap ... \sqcap R_n))^o$.
By definition of C-deletion (Definition 7), we have

$$(\gamma(L_1 \sqcap ... \sqcap L_m))^o = \gamma(L_1)^o \sqcap ... \sqcap \gamma(L_m)^o \tag{3.9}$$

By condition 2 and (3.7), we have

$$\begin{aligned} \gamma(L_1)^o \sqcap ... \sqcap \gamma(L_m)^o &= \gamma^o(L_1) \sqcap ... \sqcap \gamma^o(L_m) \\ &= \gamma^o(L_1 \sqcap ... \sqcap L_m) \end{aligned} \tag{3.10}$$

By (3.9) and (3.10), we have

$$L^o = (\gamma(L_1 \sqcap ... \sqcap L_m))^o = \gamma^o(L_1 \sqcap ... \sqcap L_m) \tag{3.11}$$

Similarly, we have

$$R^o = (\gamma(R_1 \sqcap ... \sqcap R_n))^o = \gamma^o(R_1 \sqcap ... \sqcap R_n) \tag{3.12}$$

By (3.11) and (3.12), to show that γ^o is a unifier of Γ, it is enough to prove that $L^o \equiv R^o$.
Without loss of generality, we show that $L^o \sqsubseteq R^o$. By Corollary 1, it is enough to show that for every j, $1 \leq j \leq \nu$ such that B^o_j is defined, there exists an i, $1 \leq i \leq \mu$ such that A^o_i is defined and $A^o_i \sqsubseteq B^o_j$. Since $L \sqsubseteq R$, we know that for every j, $1 \leq j \leq \nu$, there exists an i, $1 \leq i \leq \mu$ such that $A_i \sqsubseteq B_j$. On the other hand, by Lemma 7, if B^o_j is defined, then $A_i \sqsubseteq B_j$ implies that A^o_i is defined and $A^o_i \sqsubseteq B^o_j$. It shows that $L^o \sqsubseteq R^o$.
Thus, γ^o is a unifier of Γ. □

In the next lemma, we show that C-defined substitutions are unifers. Moreover, under specific conditions, they are smaller than original ones.

Lemma 14. *Let Γ be a flat $\mathcal{EL}$-top unification problem, γ a reduced ground unifier of Γ and C a subatom which is not a concept constant. If the following holds:*

1. *γ' is C-defined w.r.t. (γ, C),*
2. *there is no non-variable D of Γ such that $C = \gamma(D)$,*

then γ' is a unifier of Γ. Moreover, if C is an atom of γ then $\gamma > \gamma'$.

Proof. First, we prove the following:

$$\gamma(L) \text{ is C-defined w.r.t. } C \text{ and } (\gamma(L))^{-C} = \gamma'(L) \text{ for each atom } L \text{ of } \Gamma. \qquad (3.13)$$

We have three cases:

1. $L = A$ for some concept constant A. Then by condition 2, we have $A \neq C$. Hence $\gamma(L)$ is C-defined w.r.t. C and $\gamma(L)^{-C} = A = \gamma'(L)$.
2. $L = X$ for some variable X. Then by condition 1, $\gamma(X)$ is C-defined w.r.t. C. Hence $\gamma(L)^{-C} = \gamma'(L)$.
3. $L = \exists r.X$, where X is either a concept constant or a variable.
   - If X is a concept constant, then $\gamma(L) = \exists r.X$. Moreover, by condition 2, C does not occur in $\exists r.X$. Hence $\gamma(L)$ is C-defined w.r.t. C and $\gamma(L)^{-C} = \exists r.X = \gamma'(L)$.
   - If X is a variable, then by condition 2, $\gamma(L) \neq C$. Moreover, by condition 1, $\gamma(X)$ is C-defined w.r.t. C. Hence $\gamma(X)^{-C} = \gamma'(X)$. Thus $\gamma(L)$ is C-defined w.r.t. C and $\gamma(L)^{-C} = \exists r.\gamma(X)^{-C} = \exists r.\gamma'(X) = \gamma'(\exists r.X) = \gamma'(L)$.

This completes the proof of (3.13).
By (3.13) and Lemma 13, γ' is a unifier of Γ. On the other hand, for every variable X occurring in Γ, by Lemma 9, we have $\gamma(X) \sqsubseteq \gamma'(X)$. Moreover, since C is an atom of γ, there exists a variable Y occurring in Γ such that C occurs in $\gamma(Y)$. By Lemma 8, we have $\gamma(Y) \sqsubset \gamma'(Y)$ and thus $\gamma > \gamma'$. □

3.5 Properties of minimal unifiers in $\mathcal{EL}$-top

In the following lemma, we show that each minimal reduced ground unifier is almost local.

Lemma 15. *Let Γ be a flat $\mathcal{EL}$-top unification problem and γ a minimal reduced ground unifier of Γ. If C is an atom of γ, then one of the following holds:*

(i) There is a non-variable D of Γ such that $C \equiv \gamma(D)$.

(ii) There is a non-variable D of Γ such that $C \in SubAt(\gamma(D))$.

(iii) C is incomparable to Γ.

Proof. Let C be an atom of γ. Since γ is ground, C is either a concept constant or an existential restriction. We assume that (i) does not hold for C, i.e., there is no non-variable D of γ such that $C \equiv \gamma(D)$. We need to show that either (ii) or (iii) holds. Since C is an atom, we consider two cases:

1. $C = A$, where A is a concept constant. Since there is no non-variable D of Γ such that $C \equiv \gamma(D)$, then A does not appear in Γ. Hence C is incomparable to Γ. Thus, (iii) holds.

2. $C = \exists r.C_1$. We assume that (ii) does not hold for C, i.e., there is no non-variable D of γ such that $C \in SubAt(\gamma(D))$. We now show that (iii) holds by contradiction.
Assume that C is comparable to Γ. Let B_γ the set of all atoms of γ such that $C \sqsubset B$ for all $B \in B_\gamma$ and $\hat{D} = \bigsqcap_{E \in SubAt(C)} E \sqcap \bigsqcap_{B \in B_\gamma} B$. We show that $\hat{D}$ satisfies condition of Lemma 11 and hence $\gamma^{[C/\hat{D}]}$ is a unifier of Γ smaller than γ. There are two cases to consider depending on B_γ:

   - $B_\gamma \neq \emptyset$. Then we have
   $$C \sqsubset \bigsqcap_{B \in B_\gamma} B. \tag{3.14}$$
   On the other hand, for each $E \in SubAt(C)$, by Lemma 5, we have $C \sqsubseteq E$. Moreover, if $C = E$, then C is a subatom and thus there is no B such that $C \sqsubset B$. It yields that $B_\gamma = \emptyset$ which contradicts our assumption on non-emptiness of B_γ. Thus we must have $C \sqsubset E$ for all $E \in SubAt(C)$. It yields that
   $$C \sqsubset \bigsqcap_{E \in SubAt(C)} E. \tag{3.15}$$
   By (3.14) and (3.15), we have $C \sqsubset \hat{D}$.

   - $B_\gamma = \emptyset$. If C is a subatom, then since C is comparable, $B_\gamma = \emptyset$ and (i) does not hold for C, there is a non-variable D of Γ such that $\gamma(D) \sqsubset C$ which implies that $C \in SubAt(\gamma(D))$. It contradicts our assumption that (ii) does not hold for C. Thus C is not a subatom which implies that $C \sqsubset \hat{D}$.

In all cases, we have $C \sqsubset \hat{D}$ and then $\gamma \succ \gamma^{[C/\hat{D}]}$. Moreover, we have the following:

- If $C \sqsubseteq \gamma(D)$ for some non-variable D of Γ, then since (i) does not hold for C, we have $C \sqsubset \gamma(D)$ and thus there is an atom $B \in B_\gamma$ such that $B = \gamma(D)$, which implies that $\hat{D} \sqsubseteq \gamma(D)$.

- If $C \sqsubset B$ for some atom B of γ, then $B \in B_\gamma$. Thus, $\hat{D} \sqsubseteq B$.

By Lemma 11, $\gamma^{[C/\hat{D}]}$ is a unifier of Γ. But then it contradicts our assumption on minimality of γ. Thus, C is incomparable to Γ, i.e., (iii) holds. □

In order to get rid of incomparable atoms in a minimal unifier, we show that under specific conditions, each incomparable atom can be replaced by a concept constant such that the new substitution obtained by the replacement is also a unifier.

Lemma 16. *Let Γ be a flat $\mathcal{EL}$-top unification problem and γ a minimal reduced ground unifier of Γ. Let C be an atom of γ and $\hat{D}$ an $\mathcal{EL}$-top concept term. If the following holds:*

(i) $\hat{D} = A$ for some concept constant A,

(ii) $C \not\sqsubset B$ for all atom B of γ,

(iii) C is incomparable to Γ.

Then $\gamma^{[C/\hat{D}]}$ is a unifier of Γ.

Proof. First, we show that $A \sqsubseteq B$ implies that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$, where each A, B is either an atom of γ or equal to $\gamma(D)$ for some non-variable D of Γ. We consider the following cases:

1. Assume that $A =_{AC} C$, then $A^{[C/\hat{D}]} = \hat{D}$.
   If B is an atom of γ, then we have $C \not\sqsubset B$ which together with $C \sqsubseteq B$ implies that $C = B$. Thus $A^{[C/\hat{D}]} = \hat{D} = B^{[C/\hat{D}]}$ which implies that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$. Otherwise, $B = \gamma(D)$ for some non-variable D of Γ which contradicts our assumption on C.

2. We consider $A \neq_{AC} C$. Since A is an atom, A is either a concept constant or an existential restriction. If A is a concept constant, then $A \sqsubseteq B$ implies that $A = B$ and thus $A^{[C/\hat{D}]} = B^{[C/\hat{D}]}$. In particular, $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$. Otherwise, $A = \exists s.E$ and C occurs (modulo AC) in E. Then $A \sqsubseteq B$ yields that $B = \exists s.F$ with $E \sqsubseteq F$. Since E and F are conjunctions of atoms of γ, we have $E = E_1 \sqcap ... \sqcap E_p$ and $F = F_1 \sqcap ... \sqcap F_q$. Then $E \sqsubseteq F$ implies that for each $u, 1 \leq u \leq q$, there exists $v, 1 \leq v \leq p$ such that $E_v \sqsubseteq F_u$. By induction, we assume that $E_v \sqsubseteq F_u$ implies that $E_v^{[C/\hat{D}]} \sqsubseteq F_u^{[C/\hat{D}]}$. Thus, for each $u, 1 \leq u \leq q$, there exists $v, 1 \leq v \leq p$ such that

$$E_v^{[C/\hat{D}]} \sqsubseteq F_u^{[C/\hat{D}]}. \tag{3.16}$$

   (a) If $B \neq_{AC} C$, then $A^{[C/\hat{D}]} = \exists s.(E_1^{[C/\hat{D}]} \sqcap ... \sqcap E_p^{[C/\hat{D}]})$ and $B^{[C/\hat{D}]} = \exists s.(F_1^{[C/\hat{D}]} \sqcap ... \sqcap F_q^{[C/\hat{D}]})$ along with (3.16) yield $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

(b) If $B =_{AC} C$, then C can not occur (modulo AC) in any of $F_1, ..., F_q$ (otherwise, role depth of B is strictly larger than role depth of C). Thus, we have $B = \exists s.(F_1^{[C/\hat{D}]} \sqcap ... \sqcap F_q^{[C/\hat{D}]})$. By (3.16), we have $A^{[C/\hat{D}]} \sqsubseteq B$. Since γ is a minimal reduced ground unifier, by Lemma 15, there are three cases:

   i. If $A = \gamma(D)$ for some non-variable D of Γ, then $\gamma(D) \sqsubseteq C$ which implies that C is comparable to Γ. It contradicts (iii).

   ii. If A is a subatom, then $A \sqsubseteq C$ implies that $A = C$ and thus $A^{[C/\hat{D}]} = \hat{D}$ which implies that $A \sqsubseteq B^{[C/\hat{D}]}$.

   iii. If A is incomparable to Γ, then by (ii), $A \sqsubseteq C$ implies that $A = C$. Again, we have $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

By Lemma 10, $\gamma^{[C/\hat{D}]}$ is a unifier of Γ. □

3.6 Local unifiers

Local unifers have a structure that can be completely described by the atoms of the unification problem.

Definition 11. (Local unifiers)
*Let Γ be a solvable flat $\mathcal{EL}$-top unification problem. A reduced ground unifier γ of Γ is a **local unifier** iff for every variable X occurring in Γ, there are $n \geq 0, m \geq 0, l \geq 0$, $n+m+l \geq 1$ such that $\gamma(X) = A_1 \sqcap ... \sqcap A_n \sqcap \gamma(\exists r_1.X_1) \sqcap ... \sqcap \gamma(\exists r_m.X_m) \sqcap B_{m+1} \sqcap ... \sqcap B_{m+l}$, where:*

- $\{A_1, ..., A_n\} \subseteq Cons(\Gamma)$,
- $\exists r_1.X_1, ..., \exists r_m.X_m, \exists r_{m+1}.X_{m+1}, ..., \exists r_{m+l}.X_{m+l}$ *are non-variables of* Γ,
- *For all i, $1 \leq i \leq l$, $B_{m+i} \in SubAt(\gamma(\exists r_{m+i}.X_{m+i}))$ and $B_{m+i} \not\sqsubseteq \gamma(D)$ for each non-variable D of* Γ.

We denote:

- $S_1(X) = \{A_1, ..., A_n\}$,
- $S_2(X) = \{\exists r_1.X_1, ..., \exists r_m.X_m\}$,
- $S_3(X) = \{\exists r_{m+1}.X_{m+1}, ..., \exists r_{m+l}.X_{m+l}\}$, *where $S_3(X)$ is a multiset.*
- $Sub_\gamma(X) = \{B_{m+1}, ..., B_{m+l}\}$.

*The sets $S_1(X)$, $S_2(X)$ and the multiset $S_3(X)$ are called **local sets** of X w.r.t. γ. We denote $\overline{S_3}(X)$ the underlying set of elements of $S_3(X)$.*

In the following lemma, we show that, w.r.t. a local unifier γ of Γ, if a subatom B which is not a concept constant, has the property that there exists a non-variable D of Γ such that $\gamma(D) \sqsubseteq B$, then all subterms of B have the same property.

Lemma 17. *Let Γ be a solvable flat $\mathcal{EL}$-top unification problem and γ a local unifier of Γ. Then $\gamma(D) \sqsubseteq B$ implies that there exists a non-variable D' of Γ such that $\gamma(D') \sqsubseteq B'$, where B is a subatom such that $B = \exists r.B'$ and D is a non-variable of Γ.*

Proof. Since $\gamma(D) \sqsubseteq B = \exists r.B'$, we have $D = \exists r.X$. If X is a concept constant, then $B' = X$ and thus $D' = X$. Hence the lemma is satisfied. Otherwise, X is a variable.
Let:

- $S_2(X) = \{\exists r_1.X_1, ..., \exists r_m.X_m\}$ be a local set for X w.r.t. γ and
- $Sub_\gamma(X) = \{B_1, ..., B_l\}$.

where $B_1, ..., B_l$ are subatoms and $\exists r_1.X_1, ..., \exists r_m.X_m$ are non-variables of Γ. Thus $\gamma(D) \sqsubseteq B$ implies that $\gamma(X) \sqsubseteq B'$. Since B is a subatom, so is B'. Thus $\gamma(X) \sqsubseteq B'$ yields that either there is a non-variable $D' \in S_2(X)$ such that $\gamma(D') \sqsubseteq B'$, or there is an i, $1 \leq i \leq l$ such that $B_i \sqsubseteq B'$. If $\gamma(D') \sqsubseteq B'$ then it is obvious that the claim is true. Otherwise, since B_i is a subatom, $B_i \sqsubseteq B'$ implies that $B_i = B'$. By definition of $S_2(X)$ (Definition 11), there is a non-variable $\exists r_i.X_i$ of Γ such that $B_i \in SubAt(\gamma(\exists r_i.X_i))$. Since $B_i = B'$, we have $B' \in SubAt(\gamma(\exists r_i.X_i))$ and thus we take $D' = \exists r_i.X_i$. □

In the next lemma, we show that each atom of a local unifier has local property, i.e., it can be constructed from elements in the goal.

Lemma 18. *Let Γ be a solvable flat $\mathcal{EL}$-top unification problem and γ a local unifier of Γ. Then for every atom C of γ, the following holds:*

- *C is a concept constant, or*
- *$C \equiv \gamma(D)$ for some non-variable D of Γ, or*
- *$C \in SubAt(\gamma(D))$ for some non-variable D of Γ.*

Proof. It is enough to prove that for every variable X occurring in Γ, $C \in At(\gamma(X))$ implies that either C is a concept constant, or $C \equiv \gamma(D)$, or $C \in SubAt(\gamma(D))$, where D is a non-variable of Γ.
Let X be an arbitrary variable occurring in Γ and $C \in At(\gamma(X))$.
Let:

- $S_1(X)$ and $S_2(X)$ be local sets for X w.r.t. γ, where $S_2(X) = \{\exists r_1.X_1, ..., \exists r_m.X_m\}$.
- $Sub_\gamma(X) = \{B_1, ..., B_l\}$.

where $B_1, ..., B_l$ are subatoms and $\exists r_1.X_1, ..., \exists r_m.X_m$ are non-variables of Γ.
We prove the lemma by induction on the order of X w.r.t. the dependency order $>$ (Definition 10).

- If X is a minimal variable, then for all i, $1 \leq i \leq m$, X_i is a concept constant. We consider three cases:

  1. If $C \in S_1(X)$, then C is a concept constant.
  2. If $C \in S_2(X)$, then since $S_2(X)$ does not contain variables, C is either a concept constant or $C = \gamma(D)$ for some $D \in S_2(X)$.
  3. If $C \in At(B_i)$ for some i, $1 \leq i \leq l$, then by Lemma 17, there exists a non-variable D of γ such that $\gamma(D) \sqsubseteq C$ and thus $C \in SubAt(\gamma(D))$.

- If X is not a minimal variable, then we assume that for every variable Y with $X > Y$, $C \in At(\gamma(Y))$ implies that C satisfies the lemma. Since $C \in At(\gamma(X))$, either C is a concept constant, $C \in At(\gamma(D))$ for some non-variable $D \in S_2(X)$ or $C \in At(B_i)$ for some i, $1 \leq i \leq l$.
  By 1 and 3, it is enough to consider the case that $C \in At(\gamma(D))$. Let $D = \exists r.Z$. If Z is a concept constant, then either $C = \gamma(D)$ or C is a concept constant. Otherwise, Z is a variable and $X > Z$. If $C = \gamma(D)$ then C satisfies the lemma. Otherwise, $C \in At(\gamma(Z))$. Since $X > Z$, by induction, C satisfies the lemma.

□

Similarly to the previous result known for $\mathcal{EL}$, the following holds for $\mathcal{EL}$-top unification.

Theorem 2. *Let Γ be a solvable flat $\mathcal{EL}$-top unification problem. Then there is a minimal local unifier of Γ.*

Proof. Since Γ is solvable, there exists a minimal reduced ground unifier γ of Γ. By Lemma 15, we know that for every atom C of γ, either:

(i) there is a non-variable D of Γ such that $C \equiv \gamma(D)$, or

(ii) there is a non-variable D of Γ such that $C \in SubAt(D)$, or

(iii) C is incomparable to Γ.

We now consider an atom C of γ which is incomparable to Γ and prove that we can obtain a new unifier of Γ by replacing C by some concept constant in $Cons(\Gamma)$. We assume that C is maximal w.r.t. $\sqsubseteq$.
First, we show that C is a subatom.

1. $C = A$ with A is a concept constant. Then it is obviously that C is a subatom.

2. If $C = \exists r.C_1$, then let $SubAt(C) = \{B_1, ..., B_n\}$ and $\hat{D} = \bigsqcap_{k=1}^{n} B_k$. We assume that $n > 1$. Thus for all $1 \leq k \leq n$, we have:

   - $C \sqsubset B_k$
   - $C \sqsubset \hat{D}$

- $\hat{D} \sqsubseteq B_k$

It is obvious that $\gamma \succ \gamma^{[C/\hat{D}]}$. On the other hand, we have:

- There is no non-variable D of Γ such that $C \sqsubseteq \gamma(D)$, since C is incomparable to Γ.
- If $C \sqsubset B$ for some atom B of γ, then since C is maximal w.r.t. $\sqsubseteq$, B is comparable to Γ. On the other hand, since C is incomparable to Γ, B must be a subatom. Moreover, since $C \sqsubset B$, by Lemma 5, we have $SubAt(B) \subseteq SubAt(C) = \{B_1, ..., B_n\}$. Since B is a subatom, $SubAt(B) = \{B\}$. Thus, $B \in \{B_1, ..., B_n\}$ implies that $\hat{D} \sqsubseteq B$.

By Lemma 11, $\gamma^{[C/\hat{D}]}$ is a unifier of Γ. Moreover, we have $\gamma \succ \gamma^{[C/\hat{D}]}$. It contradicts our assumption on minimality of γ. Thus we must have $n = 1$, i.e., C is a subatom.

Second, let $\hat{D} = A$ for some $A \in Cons(\Gamma)$. Then by Lemma 16, $\gamma^{[C/\hat{D}]}$ is a unifier of Γ.
Now we show that replacing incomparable atoms terminates. We denote:

- C_γ: the number of concept constants occurring in γ,
- V: the set of all variables occurring in Γ,
- V_γ: the set of each variable $X \in V$ such that $\gamma(X) = B_1 \sqcap ... \sqcap B_l$, where $B_1, ..., B_l$ are subatoms.

It is obvious that

$$C_\gamma \geq C_{\gamma^{[C/\hat{D}]}} \tag{3.17}$$

We assume that there exists a C-defined substitution γ' w.r.t. (γ, C). Then since C is an atom of γ, by Lemma 14, γ' is a unifier of Γ and $\gamma \succ \gamma'$. It contradicts the minimality property of γ. By Lemma 12, there must exist at least one minimal variable $Y \in V$ such that $\gamma'(Y)$ is not defined. It is easy to see that $Y \in V_\gamma$.
On the other hand, since $\gamma'(Y)$ is not defined and $Y \in V_\gamma$, C occurs in every B,

$$\text{C occurs in every } B \in SubAt(\gamma(Y)). \tag{3.18}$$

We denote:

- $S_\gamma(X) = \sum_{B \in SubAt(\gamma(X))} |B|$, where $|B|$ is the role depth of B,
- $S_\gamma = C_\gamma + \sum_{X \in V_\gamma} S_\gamma(X)$,
- $T_\gamma = |V| - |V_\gamma|$.

It is easy to see that $T_\gamma \geq 0$, $T_{\gamma^{[C/\hat{D}]}} \geq 0$. On the other hand, $X \in V_\gamma$ implies that $X \in V_{\gamma^{[C/\hat{D}]}}$ and thus we have:

$$V_\gamma \subseteq V_{\gamma^{[C/\hat{D}]}} \tag{3.19}$$

We consider two cases:

1. If $V_\gamma = V_{\gamma^{[C/\hat{D}]}}$, then $T_\gamma = T_{\gamma^{[C/\hat{D}]}}$. Moreover, we have the following observation:

   (a) $S_\gamma(X) \geq S_{\gamma^{[C/\hat{D}]}}(X)$, for each variable $X \in V_\gamma$.

   (b) If C is not a concept constant, then by (3.18), we have $S_\gamma(Y) > S_{\gamma^{[C/\hat{D}]}}(Y)$.

   (c) If C is a concept constant, then C does not occur in Γ. Thus $C_\gamma > C_{\gamma^{[C/\hat{D}]}}$.

   By (1a), (1b), (1c) and (3.17), we have:

   $$S_\gamma > S_{\gamma^{[C/\hat{D}]}} \tag{3.20}$$

2. If $V_\gamma \subset V_{\gamma^{[C/\hat{D}]}}$, then $T_\gamma > T_{\gamma^{[C/\hat{D}]}}$.

Now we denote $P_\gamma = (T_\gamma, S_\gamma)$ and thus $P_\gamma > P_{\gamma^{[C/\hat{D}]}}$. Assume that $\gamma^{[C/\hat{D}]}$ is not minimal. Let γ_1 be a minimal reduced ground unifier of Γ such that $\gamma^{[C/\hat{D}]} \succ \gamma_1$.
Since $\gamma^{[C/\hat{D}]} \succ \gamma_1$, for every variable $X \in V_{\gamma^{[C/\hat{D}]}}$, $\gamma^{[C/\hat{D}]}(X) \sqsubseteq \gamma_1(X)$ implies that $X \in V_{\gamma_1}$ and thus we have:

$$V_{\gamma^{[C/\hat{D}]}} \subseteq V_{\gamma_1} \tag{3.21}$$

- If $V_\gamma = V_{\gamma_1}$, then by (3.19) and (3.21) we have $V_\gamma = V_{\gamma^{[C/\hat{D}]}} = V_{\gamma_1}$. Thus we have $T_\gamma = T_{\gamma_1}$. Moreover, it is obvious that $C_{\gamma^{[C/\hat{D}]}} \geq C_{\gamma_1}$. On the other hand, for each variable $X \in V_\gamma$, we have $X \in V_{\gamma_1}$ and together with $\gamma^{[C/\hat{D}]}(X) \sqsubseteq \gamma_1(X)$ it yields that $S_{\gamma^{[C/\hat{D}]}}(X) \geq S_{\gamma_1}(X)$. Thus we have:

  $$S_{\gamma^{[C/\hat{D}]}} \geq S_{\gamma_1} \tag{3.22}$$

  By (3.20) and (3.22), we have $S_\gamma > S_{\gamma_1}$.

- If $V_\gamma \subset V_{\gamma_1}$, then $T_\gamma > T_{\gamma_1}$.

Thus $P_\gamma > P_{\gamma_1}$. By using the same construction, we obtain a chain of minimal unifiers $\gamma_0 = \gamma, \gamma_1, ...$ such that $P_{\gamma_i} > P_{\gamma_{i+1}}$, for all $i \geq 0$.
Since the order $>$ over P_{γ_i}, $i \geq 0$ is a well-founded order, there exists an n large enough such that γ_n does not contain any incomparable atoms. By definition of local unifiers (Definition 11), γ_n is local. Hence together with minimality of γ_n, this completes the proof of the lemma. □

3.7 An $\mathcal{EL}$-top unification problem with infinitely many minimal local unifiers

In the following example, we show that a flat $\mathcal{EL}$-top unification problem Γ can have infinitely many local unifiers. Hence locality is not strong enough to obtain a decision procedure for $\mathcal{EL}$-top unification.

Example 1. *We consider the flat $\mathcal{EL}$-top unification problem* $\Gamma := \{X \equiv Y \sqcap A, Y \sqcap \exists r.X \equiv \exists r.X, Z \sqcap \exists r.X \equiv \exists r.X\}$.
For every $n \geq 1$, we construct a substitution γ_n without top such that:

- $\gamma_n(X) = A \sqcap \exists r.A \sqcap ... \sqcap \exists r^n.A$,
- $\gamma_n(Y) = \exists r.A \sqcap ... \sqcap \exists r^n.A$,
- $\gamma_n(Z) = \exists r^{n+1}.A$

Proposition 2. *γ_n is a local unifier of Γ, for all $n \geq 1$.*

Proof. Since $Z \sqcap \exists r.X \equiv \exists r.X$, we have

$$\exists r.X \sqsubseteq Z \tag{3.23}$$

Since $Y \sqcap \exists r.X \equiv \exists r.X$, we have

$$\exists r.X \sqsubseteq Y \tag{3.24}$$

Let $D_1 = \exists r.X$, $D_2 = A$, then D_1, D_2 are non-variables in Γ. It is easy to see that γ_n is a unifier of Γ. Notice that $SubAt(\gamma(D_1)) = \{\exists r.A, ..., \exists r^{n+1}.A\}$. Thus, for all i, $1 \leq i \leq n+1$, we write $B_i := \exists r^i.A$. Thus for all $n \geq 1$, we have

$$\gamma_n(Y) = \bigsqcap_{1 \leq i \leq n} B_i$$

$$\gamma_n(Z) = B_{n+1} \tag{3.25}$$

We prove, by contradiction, that γ_n is minimal for all $n \geq 1$. Assume that there is a k, $1 \leq k \leq n$ such that γ_k is not minimal. Then there exists a unifier γ'_k such that $\gamma_k > \gamma'_k$. Thus we have $\gamma_k(Z) \sqsubseteq \gamma'_k(Z)$. Since $\gamma_k(Z) \in SubAt(\gamma_k(D_1))$, $\gamma_k(Z) \sqsubseteq \gamma'_k(Z)$ implies that $\gamma_k(Z) = \gamma'_k(Z)$. By (3.25) we have

$$\gamma'_k(Z) = B_{k+1} \tag{3.26}$$

On the other hand, by (3.23) and (3.26), we have

$$\gamma'_k(\exists r.X) \sqsubseteq \gamma'_k(Z) = B_{k+1} \tag{3.27}$$

By (3.27) and Lemma 5, we have

$$SubAt(B_{k+1}) \subseteq SubAt(\gamma'_k(\exists r.X)) \tag{3.28}$$

By (3.28), since $SubAt(B_{k+1}) = \{\exists r.B_k\}$, we have

$$B_k \in SubAt(\gamma'_k(X)) \tag{3.29}$$

Since $\gamma_k(Y) = \gamma'_k(Y)$ implies that $\gamma_k(X) = \gamma'_k(X)$, then if $\gamma_k(Y) = \gamma'_k(Y)$, we have $\gamma_k = \gamma'_k$ which contradicts $\gamma_k > \gamma'_k$. Thus we must have

$$\gamma_k(Y) \sqsubset \gamma'_k(Y) \tag{3.30}$$

We now show that in fact $B_k \notin SubAt(\gamma'_k(X))$ which contradicts (3.29).
Let:

$$\gamma'_k(Y) = B_{k_1} \sqcap ... \sqcap B_{k_p}, \tag{3.31}$$

where $B_{k_1}, ..., B_{k_p}$ are reduced and pairwise incomparable w.r.t. subsumption and $p \geq 1$. By (3.25), we have

$$\gamma_k(Y) = \bigsqcap_{1 \leq i \leq k} B_i \tag{3.32}$$

By (3.31), (3.32) and Corollary 1, (3.30) implies that for every u, $1 \leq u \leq p$, there exists an v, $1 \leq v \leq k$ such that $B_v \sqsubseteq B_{k_u}$.
Since B_v is a subatom, by Lemma 3, $B_v \sqsubseteq B_{k_u}$ implies that $B_v = B_{k_u}$.
Thus we have

$$\{B_{k_1}, ..., B_{k_p}\} \subset \{B_1, ..., B_k\},$$
$$SubAt(\gamma'_k(Y)) = \{B_{k_1}, ..., B_{k_p}\}. \tag{3.33}$$

Since $\exists r.X \sqcap Y \equiv \exists r.X$, we have $\exists r.X \sqsubseteq Y$. Hence $\gamma'_k(\exists r.X) \sqsubseteq \gamma'_k(Y)$ which together with Lemma 5 yields that

$$SubAt(\gamma'_k(Y)) \subseteq SubAt(\gamma'_k(\exists r.X)) \tag{3.34}$$

By (3.33), there exists an i, $1 \leq i \leq k$ such that

$$B_i \notin SubAt(\gamma'_k(Y)) \tag{3.35}$$

Since $\gamma'_k(X) = \gamma'_k(Y \sqcap A)$, we have

$$SubAt(\gamma'_k(X)) = SubAt(\gamma'_k(Y)) \cup \{A\} \tag{3.36}$$

Thus by (3.36), for each j, $1 \leq j \leq k$:

$$B_j \notin SubAt(\gamma'_k(Y)) \text{ iff } B_j \notin SubAt(\gamma'_k(X)). \tag{3.37}$$

By (3.34), for each j, $1 \leq j \leq k$:

$$B_j \notin SubAt(\gamma'_k(X)) \text{ implies that } B_{j+1} \notin SubAt(\gamma'_k(Y)) \tag{3.38}$$

By (3.37) and (3.38), for each j, $1 \leq j \leq k$:

$$B_j \notin SubAt(\gamma'_k(X)) \text{ implies that } B_t \notin SubAt(\gamma'_k(X)), \text{ for each } t, j \leq t \leq k. \tag{3.39}$$

By (3.35) and (3.37), we have

$$B_i \notin SubAt(\gamma'_k(X)) \tag{3.40}$$

By (3.40), (3.39) and $1 \leq i \leq k$, we have

$$B_k \notin SubAt(\gamma'_k(X)) \tag{3.41}$$

Hence (3.41) contradicts (3.29). Thus γ_n is minimal for all $n \geq 1$. By (3.25), γ_n is local and thus γ_n is a minimal local unifier, for all $n \geq 1$. □

Chapter 4

Decision procedure for $\mathcal{EL}$-top unification

4.1 Decision problem

For a given flat $\mathcal{EL}$-top unification problem Γ, we want to decide whether Γ is solvable or not. As we mentioned in the previous section, the notion of local unifiers is not strong enough to justify immediately a ”guess and then check” decision procedure, since we have seen an example with infinitely many local unifiers. To deal with this difficulty, we need to show how to reduce the unification problem even further to the problem of existence of ”small local unifiers”. We first introduce some notations, then we define *small local unifiers* formally and present the decision procedure for $\mathcal{EL}$-top unification.

4.1.1 R-trees

For each variable, we create an R-tree. The nodes are labeled with variables or concept names. Relation between two variables connected by an edge in such a tree is a generalization of the dependency relation introduced in Section 3.4.2.

Definition 12. (R-trees)
Let Γ be a flat $\mathcal{EL}$-top unification problem, γ a local unifier of Γ and $Var(\Gamma)$ the set of all variables occurring in Γ. We denote $\Sigma_\Gamma = Cons(\Gamma) \cup Var(\Gamma)$ and $\omega = \{0, 1, ..., |\Sigma_\Gamma| - 1\}$.
Let N_Γ be the set of all role names occurring in Γ, $X \in Var(\Gamma)$ a variable with local sets $S_1(X)$, $S_2(X)$ and the underlying set $\overline{S_3}(X)$ defined as follows:

- $S_1(X) = \{A_1, ..., A_n\}$,
- $S_2(X) = \{\exists r_{n+1}.X_{n+1}, ..., \exists r_{n+m}.X_{n+m}\}$,
- $\overline{S_3}(X) = \{\exists r_{n+m+1}.X_{n+m+1}, ..., \exists r_{n+m+l}.X_{n+m+l}\}$.

The arity function $\nu_\gamma : \Sigma_\Gamma \rightarrow \mathbb{N}$ is defined as follows:

- *For every concept constant A, $\nu_\gamma(A) = 0$.*
- *For every variable X, $\nu_\gamma(X) = n$ iff $|S_1(X) \cup S_2(X) \cup \overline{S_3}(X)| = n$.*

An R-tree R_X for a variable X and γ is a tuple $\{E_X, V_X\}$, where V_X, E_X are partial functions and are defined as follows:

- $V_X : \omega^* \to \Sigma_\Gamma$.
  - $\epsilon \in dom(V_X)$,
  - For all $u \in \omega^*$ and $i \in \omega$, $ui \in dom(V_X)$ iff $u \in dom(V_X)$ and $i < \nu_\gamma(V_X(u))$. A leaf of R_X is a node $u \in dom(V_X)$ with $\nu_\gamma(V_X(u)) = 0$.
  - $V_X(\epsilon) = X$,
  - $V_X(ui) = A_i \in S_1(V_X(u))$ if $i < |S_1(V_X(u))|$,
  - $V_X(ui) = X_i \in S_2(V_X(u)) \cup \overline{S_3}(V_X(u))$ if $|S_1(V_X(u))| \leq i < |S_1(V_X(u)) \cup S_2(V_X(u)) \cup \overline{S_3}(V_X(u))|$.
- $E_X : \omega^+ \to N_\Gamma \cup \{\#\}$.
  - For all $u \in \omega^+$, $u \in dom(E_X)$ iff $u \in dom(V_X)$,
  - $E_X(ui) = r_i$ if $|S_1(V_X(u))| \leq i < |S_1(V_X(u)) \cup S_2(V_X(u)) \cup \overline{S_3}(V_X(u))|$,
  - $E_X(ui) = \#$ if $i < |S_1(V_X(u))|$.

A path on an R-tree is defined as follows.

Definition 13. (R-paths)
A path d on an R-tree R is called an ***R-path*** *iff it ends up with a leaf.*

By $d(r_1, ..., r_k, A)$, we denote an R-path d on an R-tree which follows $r_1, ..., r_k$ edges in this order and ends with a leaf A, where $r_1, ..., r_k$ are the edge labels on d. We say that d represents a subatom $B = \exists r_1...r_{l_d}.A$, where $l_d = k$ if $r_k \neq \#$ and $l_d = k - 1$ if $r_k = \#$.

Example 2. *We continue with Example 1 and consider the local unifier γ, where:*

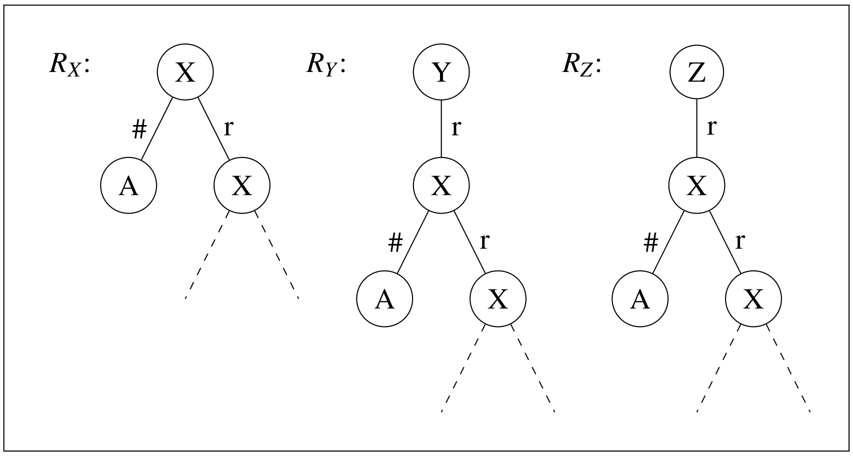
- $\gamma(X) = A \sqcap \exists r.A \sqcap \exists rr.A$
- $\gamma(Y) = \exists r.A \sqcap \exists rr.A$
- $\gamma(Z) = \exists rrr.A$.

Regarding γ, we have $\{\exists r.A, \exists rr.A, \exists rrr.A\} \subseteq SubAt(\gamma(X))$. Thus S_1, S_2 and $\overline{S_3}$ for each variable are defined as follows:

- $S_1(X) = \{A\}, S_2(X) = \emptyset, \overline{S_3}(X) = \{\exists r.X\}$,
- $S_1(Y) = \emptyset, S_2(Y) = \emptyset, \overline{S_3}(Y) = \{\exists r.X\}$,
- $S_1(Z) = \emptyset, S_2(Z) = \emptyset, \overline{S_3}(Z) = \{\exists r.X\}$.

R-trees R_X, R_Y and R_Z w.r.t. γ are illustrated in the graph 4.1:

Figure 4.1: Example of R-trees



Each subatom of $\gamma(X)$ is represented by an R-path on R_X. This is shown by the following lemma.

Lemma 19. *Let* Γ *be a flat unification problem and* γ *a local unifier of* Γ. *Then for every variable X occurring in* Γ, $B \in SubAt(\gamma(X))$ *if there exists a k,* $k \geq 1$ *and an R-path* $d(r_1, ..., r_k, A)$ *on* R_X *such that* $\exists r_1...r_{l_d}.A = B$.

Proof. We prove this lemma by induction on the depth of B.

- B is a concept constant. Then $B \in SubAt(\gamma(X))$ implies that $B \in S_1(X)$. Thus the R-path $d(\#, B)$ satisfies the lemma.

- $B = \exists r.B'$, where B' is a subatom. Then $B \in SubAt(\gamma(X))$ implies that $B \in SubAt(\gamma(\exists r.Y))$, where $\exists r.Y \in S_2(X) \cup \overline{S_3}(X)$. If Y is a concept constant, then the R-path $d(r, B')$ satisfies the lemma. Otherwise, Y is a variable. Moreover, since $B \in SubAt(\gamma(\exists r.Y))$, we have $B' \in SubAt(\gamma(Y))$. Thus by induction, there is an R-path $d'(r_1, ..., r_k, A)$ on R_Y such that $B' = \exists r_1...r_{l_{d'}}.A$. Thus the R-path $d(r, r_1, ..., r_k, A)$ on R_X satisfies the lemma.

□

4.1.2 Small local unifiers

Given a flat $\mathcal{EL}$-top unification problem Γ, we show that the solvability of Γ can be reduced to the problem of the existence of a *small local unifier* γ which has $|Sub_\gamma(X)|$ bounded for all variables X occurring in Γ.
The following lemma shows that under a specific condition, a subsitution obtained by replacing one subatom by another in a local unifier is also a unifier. In this way,

we can reduce number of subatoms and obtain a small local unifier which will be shown in Lemma 22.

Lemma 20. *Let Γ be a flat $\mathcal{EL}$-top unification problem and γ a local unifier of Γ. Let C be an atom of γ and $\hat{D}$ an $\mathcal{EL}$-top concept term. If the following holds:*

1. *$C, \hat{D}$ are subatoms and $\hat{D}$ does not contain C,*
2. *For every non-variable D of Γ, $\gamma(D) \sqsubseteq C$ implies that $\gamma(D) \sqsubseteq \hat{D}$.*

Then $\gamma^{[C/\hat{D}]}$ is a unifier of Γ.

Proof. First, we show that $A \sqsubseteq B$ implies that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$, where each A, B is either an atom of γ or equal to $\gamma(D)$ for some non-variable D of Γ. We consider the following cases:

1. Assume that $A =_{AC} C$, then $A^{[C/\hat{D}]} = \hat{D}$. Since C is a subatom, $C \sqsubseteq B$ implies that $C = B$. Thus we have $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.
2. We consider $A \neq_{AC} C$. Since A is an atom, A is either a concept constant or an existential restriction. If A is a concept constant, then $A \sqsubseteq B$ implies that $A = B$ and thus $A^{[C/\hat{D}]} = B^{[C/\hat{D}]}$. In particular, $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$. Otherwise, $A = \exists s.E$ and C occurs (modulo AC) in E. Then $A \sqsubseteq B$ yields that $B = \exists s.F$ with $E \sqsubseteq F$. Since E and F are conjunctions of atoms of γ, we have $E = E_1 \sqcap ... \sqcap E_p$ and $F = F_1 \sqcap ... \sqcap F_q$. Then $E \sqsubseteq F$ implies that for each $u, 1 \leq u \leq q$, there exists $v, 1 \leq v \leq p$ such that $E_v \sqsubseteq F_u$. By induction, we assume that $E_v \sqsubseteq F_u$ implies that $E_v^{[C/\hat{D}]} \sqsubseteq F_u^{[C/\hat{D}]}$. Thus, for each $u, 1 \leq u \leq q$, there exists $v, 1 \leq v \leq p$ such that
$$E_v^{[C/\hat{D}]} \sqsubseteq F_u^{[C/\hat{D}]}. \tag{4.1}$$
   - If $B \neq_{AC} C$, then $A^{[C/\hat{D}]} = \exists s.(E_1^{[C/\hat{D}]} \sqcap ... \sqcap E_p^{[C/\hat{D}]})$ and $B^{[C/\hat{D}]} = \exists s.(F_1^{[C/\hat{D}]} \sqcap ... \sqcap F_q^{[C/\hat{D}]})$ along with (4.1) yield $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.
   - If $B =_{AC} C$, then C can not occur (modulo AC) in any of $F_1, ..., F_q$ (otherwise, role depth of B is strictly larger than role depth of C). Thus, we have $B = \exists s.(F_1^{[C/\hat{D}]} \sqcap ... \sqcap F_q^{[C/\hat{D}]})$. By (4.1), we obtain $A^{[C/\hat{D}]} \sqsubseteq B$. By Lemma 18, A is either a subatom, or equal to $\gamma(D)$ for some non-variable D of Γ. If $A = \gamma(D)$, then by condition (2), $A \sqsubseteq C$ implies that $A \sqsubseteq \hat{D}$. By condition 1 and Lemma 6, we have $A^{[C/\hat{D}]} \sqsubseteq \hat{D} = B^{[C/\hat{D}]}$. Otherwise, A is a subatom and thus $A \sqsubseteq B$ implies that $A = B$. It yields that $A^{[C/\hat{D}]} \sqsubseteq B^{[C/\hat{D}]}$.

By Lemma 10, $\gamma^{[C/\hat{D}]}$ is a unifier of Γ. □

In the following lemma, we show that under certain conditions, specific subsumptions are preserved w.r.t. the substitution γ' which is obtained from γ by applying replacement. We use this lemma to show that reducing number of subatoms preserves locality in Lemma 22.

Lemma 21. *Let Γ be a solvable flat $\mathcal{EL}$-top unification problem, γ a local unifier of Γ, C and C' subatoms such that:*

1. *C is not a concept constant,*
2. *C' does not contain C,*
3. *for every non-variable D of Γ, $\gamma(D) \sqsubseteq C$ implies that $\gamma(D) \sqsubseteq C'$.*

Then for every subatom B which is an atom of γ and non-variable D of Γ, $\gamma(D) \sqsubseteq B$ implies that $\gamma^{[C/C']}(D) \sqsubseteq B^{[C/C']}$.

Proof. Let D be a non-variable of Γ such that $\gamma(D) \sqsubseteq B$. We have to show that

$$\gamma^{[C/C']}(D) \sqsubseteq B^{[C/C']} \tag{4.2}$$

First, we consider the following cases:

(a) If B does not contain C, then by Lemma 6, we have

$$\gamma^{[C/C']}(D) \sqsubseteq B = B^{[C/C']} \tag{4.3}$$

Hence by (4.3), (4.2) is satisfied.

(b) If $B = C$, then $B^{[C/C']} = C'$. By condition 3, we have $\gamma(D) \sqsubseteq C'$. Since C' does not contain C, by Lemma 6, it yields that

$$\gamma^{[C/C']}(D) \sqsubseteq C' \tag{4.4}$$

and thus we have

$$\gamma^{[C/C']}(D) \sqsubseteq B^{[C/C']} \tag{4.5}$$

Hence by (4.5), (4.2) is satisfied.

Second, we consider the case that B contains C. Thus the role depth of B is larger or equal to that of C. We prove the lemma by induction on the role depth of B.

- If the role depth of B is equal to that of C, then since B contains C, we have $B = C$ and thus by case (b), (4.2) is satisfied.
- If the role depth of B greater than that of C, then we assume that $B = \exists r.B'$ for some subatom B'. It is obvious that $B \neq C$. Since B contains C and $B \neq C$, then B' contains C. Thus we have

  $$B^{[C/C']} = \exists r.B'^{[C/C']} \tag{4.6}$$

  If $D = A$ for some concept constant A, then $\gamma(D) \sqsubseteq B$ implies that $A = B$ which is impossible. Hence D is not a concept constant. Let $D = \exists r.X$ for some non-variable $\exists r.X$ of Γ, then $\gamma(D) \sqsubseteq B$ implies that $\gamma(X) \sqsubseteq B'$. If B' is a concept constant, then since B' contains C, we must have $B' = C$. Thus C

is a concept constant which contradicts our assumption on C (1). Hence B' is not a concept constant.

Let $S_1(X)$, $S_2(X)$ be local sets of X w.r.t. γ and $Sub_\gamma(X) = \{B_1, ..., B_l\}$. By Corollary 1, $\gamma(X) \sqsubseteq B'$ implies that either there exists a non-variable $D' \in S_1(X) \cup S_2(X)$ such that $\gamma(D') \sqsubseteq B'$, or $B_i \sqsubseteq B'$ for some i, $1 \leq i \leq l$. However, if $D' \in S_1(X)$, i.e., D' is a concept constant, then since B' is a subatom, by Lemma 3, $\gamma(D') = D' \sqsubseteq B'$ implies that $D' = B'$. It means that B' is a concept constant which is a contradiction. Hence $D' \in S_2(X)$. We consider two cases:

(i) If $\gamma(D') \sqsubseteq B'$, then by induction, we have

$$\gamma^{[C/C']}(D') \sqsubseteq B'^{[C/C']} \tag{4.7}$$

Since $D' \in S_2(X)$, we have

$$\gamma^{[C/C']}(X) \sqsubseteq \gamma^{[C/C']}(D') \tag{4.8}$$

By (4.7) and (4.8), we have

$$\gamma^{[C/C']}(X) \sqsubseteq B'^{[C/C']} \tag{4.9}$$

By (4.9), it yields that

$$\gamma^{[C/C']}(D) = \gamma^{[C/C']}(\exists r.X) = \exists r.\gamma^{[C/C']}(X) \sqsubseteq \exists r.B'^{[C/C']} \tag{4.10}$$

By (4.6) and (4.10), we have

$$\gamma^{[C/C']}(D) \sqsubseteq B^{[C/C']} \tag{4.11}$$

Hence by (4.11), (4.2) is satisfied.

(ii) If $B_i \sqsubseteq B'$ for some i, $1 \leq i \leq l$, then since B_i is a subatom, by Lemma 3, we have

$$B_i = B' \tag{4.12}$$

On the other hand, since $B_i \in Sub_\gamma(X)$, we have

$$\gamma^{[C/C']}(X) \sqsubseteq B_i^{[C/C']} \tag{4.13}$$

By (4.12) and (4.13), we have

$$\gamma^{[C/C']}(X) \sqsubseteq B'^{[C/C']} \tag{4.14}$$

which yields that

$$\gamma^{[C/C']}(D) = \exists r.\gamma^{[C/C']}(X) \sqsubseteq \exists r.B'^{[C/C']} \tag{4.15}$$

By (4.6) and (4.15), we have

$$\gamma^{[C/C']}(D) \sqsubseteq B^{[C/C']} \tag{4.16}$$

Hence by (4.16), (4.2) is satisfied.

In all cases, (4.2) is satisfied.

□

The following result shows that if a local unifier has too many subatoms which are also atoms of the unifier, then we can construct a new one which has fewer subatoms.

Lemma 22. *Let* Γ *be a solvable flat* $\mathcal{EL}$*-top unification problem and* S *the set of all non-variables occurring in* Γ*. Then there exists a local unifier* γ' *such that for every variable* X *in* Γ*,* $|Sub_{\gamma'}(X)| \leq 2^{|S|}$.

Proof. Since Γ is solvable, by Theorem 2, there exists a local unifier γ of Γ. Let X be a variable occurring in Γ and $Sub_{\gamma}(X) = \{C_1, ..., C_l\}$.
For every i, $1 \leq i \leq l$, let $D(C_i) = \{D \in S \mid C_i \in SubAt(\gamma(D))\}$. If $l \leq 2^{|S|}$ then we take $\gamma' = \gamma$ which satisfies the lemma. Otherwise, we assume that $l > 2^{|S|}$. Thus there exists i, j, $1 \leq i < j \leq l$ such that

$$D(C_i) = D(C_j) \tag{4.17}$$

Without loss of generality, we assume that the role depth of C_i is larger or equal to that of C_j. Then it is obvious that

1. C_j does not contain C_i,
2. C_i is not a concept constant.

By Lemma 20, $\gamma^{[C_i/C_j]}$ is a unifier of Γ.
Moreover, by (4.17), for every non variable D of Γ, $\gamma(D) \sqsubseteq C_i$ implies that $\gamma(D) \sqsubseteq C_j$. Hence γ satisfies Lemma 21. Now we show that $\gamma^{[C_i/C_j]}$ is a local unifier.
Let B be an atom of $\gamma^{[C_i/C_j]}$. We need to show that one of the following holds.

(i) B is a concept constant,

(ii) $B = \gamma^{[C_i/C_j]}(D)$ for some non-variale D of Γ,

(iii) $B \in SubAt(\gamma^{[C_i/C_j]}(D))$ for some non-variale D of Γ.

Since B is an atom of $\gamma^{[C_i/C_j]}$, there is an atom C of γ such that $C^{[C_i/C_j]} = B$. Since γ is a local unifier of Γ, there are only three cases:

(a) C is a concept constant. Then since $C^{[C_i/C_j]} = B$, B is a concept constant. Hence (i) holds.

(b) $C = \gamma(D)$ for some non-variable D of Γ. Then $C^{[C_i/C_j]} = \gamma^{[C_i/C_j]}(D)$. Hence $B = \gamma^{[C_i/C_j]}(D)$. It means that (ii) holds.

(c) $C \in SubAt(\gamma(D))$. Then we have $\gamma(D) \sqsubseteq C$. Since γ satisfies Lemma 21, $\gamma(D) \sqsubseteq C$ implies that $\gamma^{[C_i/C_j]}(D) \sqsubseteq C^{[C_i/C_j]}$. On the other hand, since C is a subatom, so is $C^{[C_i/C_j]}$. It yields that $B \in SubAt(\gamma^{[C_i/C_j]}(D))$. Hence (iii) holds.

Thus $\gamma^{[C_i/C_j]}$ is a local unifier.
Moreover, since γ is a local unifier of Γ, for every variable Y occurring in Γ, we know that $\gamma(Y) = A_1 \sqcap ... \sqcap A_n \sqcap \gamma(D_1) \sqcap ... \sqcap \gamma(D_m) \sqcap B_1 \sqcap ... \sqcap B_l$, where:

- $\{A_1, ..., A_n\} \subseteq Cons(\Gamma)$,
- $\{D_1, ..., D_m\} \subseteq S$ and $\{D_1, ..., D_m\} \cap Cons(\Gamma) = \emptyset$,
- $Sub_\gamma(X) = \{B_1, ..., B_l\}$.

Since C_j is not a concept constant, we have

$$\gamma^{[C_i/C_j]}(Y) = A_1 \sqcap ... \sqcap A_n \sqcap \bigsqcap_{1 \le k \le m} \gamma^{[C_i/C_j]}(D_k) \sqcap B_1^{[C_i/C_j]} \sqcap ... \sqcap B_l^{[C_i/C_j]} \tag{4.18}$$

By (4.18), we have

$$Sub_{\gamma^{[C_i/C_j]}}(Y) \subseteq \{B_1^{[C_i/C_j]}, ..., B_l^{[C_i/C_j]}\} \tag{4.19}$$

which implies that

$$|Sub_{\gamma^{[C_i/C_j]}}(Y)| \le l = |Sub_\gamma(Y)| \tag{4.20}$$

On the other hand, since $\{C_i, C_j\} \subseteq Sub_\gamma(X)$, by (4.20), we have

$$|Sub_{\gamma^{[C_i/C_j]}}(X)| < |Sub_\gamma(X)| \tag{4.21}$$

Let:

1. $\gamma_0 = \gamma$,
2. $\gamma_1 = \gamma^{[C_i/C_j]}$,
3. $S_\gamma = \sum_{X \in V} |Sub_\gamma(X)|$, where V is the set of all variables occurring in Γ.

By (4.20) and (4.21), we have

$$S_{\gamma_0} > S_{\gamma_1} \tag{4.22}$$

By using the same construction, we obtain a chain of local unifiers $\gamma_0, \gamma_1, ...$ such that $S_{\gamma_i} > S_{\gamma_{i+1}}$, for all $i \ge 0$.
Since the order $>$ over S_{γ_i}, $i \ge 0$ is a well-founded order, there exists an $n \ge 1$ large enough such that $\gamma' = \gamma_n$ satisfies the lemma. □

The unifier γ' of Γ which satisfies Lemma 22 is called **small local unifier**. We now define it formally as follows.

Definition 14. (Small local unifiers)
*Let Γ be a flat $\mathcal{EL}$-top unification problem, γ a local unifier of Γ and S the set of all non-variables occurring in Γ. Then γ is called **small** iff for every variable X occurring in Γ, we have $|Sub_\gamma(X)| \le 2^{|S|}$.*

By Theorem 2 and Lemma 22, we have the following theorem.

Theorem 3. *Let* Γ *be a flat* $\mathcal{EL}$*-top unification problem. Then* Γ *is solvable iff it has a small local unifier.*

Next, we define a dependency path such that along the path, each variable depends on previous variables w.r.t. dependency relation (Definition 10).

Definition 15. (Dependency path)
Let Γ *be a flat* $\mathcal{EL}$*-top unification problem,* γ *a local unifier of* Γ*. For each variable* X *occurring in* Γ*, a dependency path starting from* X *is a chain* $X > Y_1 > ... > Y_{n-1}$*, where* $Y_1, ..., Y_{n-1}$ *are variables occurring in* Γ *and* $>$ *is the dependency order. We also say that the length of this path is* n*.*

For each variable X occurring in Γ, we define $N_\gamma(X)$ as the length of the longest dependency path starting from X.
For a small local unifier, we can restrict size of each solution and height of sub-atoms used in the solution. We do it in the next two lemmas.

Lemma 23. *Let* Γ *be a flat unification problem,* γ *a small local unifier of* Γ*,* S *the set of all non-variables and* V *the set of all variables in* Γ*. Let* $v = |V|$ *and* $m = |S|$*. Then for every variable* X *in* Γ *we have* $|SubAt(\gamma(X))| < m^v(1 + 2^{m+1})$*.*

Proof. Let X be a variable occurring in Γ, $n = |Cons(\Gamma)|$ and $l = 2^m$. First, we show that

$$|SubAt(\gamma(X))| \leq m^u + (n + l)(m^{u-1} + m^{u-2} + ... + 1), \tag{4.23}$$

where $u = N_\gamma(X)$. It is obvious that $S_2(X) \subseteq S$ and thus we have

$$|S_2(X)| \leq |S| = m \tag{4.24}$$

Moreover, since $S_1(X) \subseteq Cons(\Gamma)$, we have

$$|S_1(X)| \leq |Cons(\Gamma)| = n \tag{4.25}$$

We prove (4.23) by induction on the order of X w.r.t. the dependency order $>$.

- If X is minimal, then we have $u = 1$ and

$$|SubAt(\gamma(X))| = |S_1(X)| + |S_2(X)| + |Sub_\gamma(X)| \tag{4.26}$$

  By Lemma 22, we have

$$|Sub_\gamma(X)| \leq l \tag{4.27}$$

  Thus by (4.26), (4.24), (4.25) and (4.27), we have

$$|SubAt(\gamma(X))| \leq n + m + l \tag{4.28}$$

  Since $u = 1$, by (4.28), (4.23) is satisfied.

- If X is not minimal, then for each variable Y occurring in $S_2(X)$, we have

$$N_\gamma(Y) = n_Y \leq u - 1 \quad (4.29)$$

On the other hand, by induction we have

$$|SubAt(\gamma(Y))| \leq m^{n_Y} + (n + l)(m^{n_Y - 1} + m^{n_Y - 2} + ... + 1) \quad (4.30)$$

By (4.29) and (4.30), we have

$$|SubAt(\gamma(Y))| \leq m^{u-1} + (n + l)(m^{u-2} + m^{u-3} + ... + 1) \quad (4.31)$$

By (4.25), (4.31) and (4.27), we have

$|SubAt(\gamma(X))| \leq |S_1(X)| + \sum_{Y \in S_2(X)} |SubAt(\gamma(Y))| + |Sub_\gamma(X)| \leq$

$n + m \times (m^{u-1} + (n + l)(m^{u-2} + m^{u-3} + ... + 1)) + l =$

$m^u + (n + l)(m^{u-1} + m^{u-2} + ... + 1).$

This completes the proof of (3.14).
Since $u \leq v$, we have

$$|SubAt(\gamma(X))| \leq m^v + (n + l) \times m^v = m^v(1 + n + l) \quad (4.32)$$

On the other hand, we have

$$n \leq m < 2^m \quad (4.33)$$

By (4.32) and (4.33), we have

$$|SubAt(\gamma(X))| < m^v(1 + 2^m + 2^m) = m^v(1 + 2^{m+1})$$

□

In the next lemma, we show that w.r.t. a small local unifier γ, an R-path that represents a subatom of $\gamma(X)$ is of at most exponential size.

Lemma 24. *Let Γ be a flat $\mathcal{EL}$-top unification problem, S the set of all non-variables, V the set of all variables in Γ and γ a small local unifier of Γ. We denote $v = |V|$, $m = |S|$. Then for every variable X occurring in Γ, if $B \in SubAt(\gamma(X))$ and there exists an R-path $d(r_1, ..., r_k, A)$ on R_X such that $\exists r_1...r_{l_d}.A = B$, then $k \leq v \times m^v(1 + 2^{m+1})$.*

Proof. Let X be an arbitrary variable occurring in Γ. We assume that $B \in SubAt(\gamma(X))$. By Lemma 19, there exists a k, $k \geq 1$, and an R-path $d(r_1, ..., r_k, A)$ on R_X such that $\exists r_1...r_{l_d}.A = B$. We assume that $k > v \times m^v(1 + 2^{m+1})$. Then there exists a variable Y such that Y occurs at least $m^v(1 + 2^{m+1})$ times on the R-path. Let p_1 be the position of the root and $p_{y_1}, ..., p_{y_n}$ the positions where Y occurs with $1 \le y_1 < y_2 < ... < y_n$ and $n = m^v(1 + 2^{m+1})$. Thus for each i, $1 \le i \le n$, $d_i = d(r_{y_i}, r_{y_i+1}, ..., r_k, A)$ is an R-path on R_Y ending with a concept constant A. Since $B \in SubAt(\gamma(X))$, each d_i represents a subatom B_i such that $B_i \in SubAt(\gamma(Y))$. Thus $|SubAt(\gamma(Y))| \ge n = m^v(1 + 2^{m+1})$ which contradicts Lemma 23. It means that $k \le v \times m^v(1 + 2^{m+1})$. □

4.2 Algorithm

For the sake of better readability, we divide our decision algorithm into two parts: the proccedure *guess_subatom* and the main algorithm that uses it.
Let Γ be a flat $\mathcal{EL}$-top unification problem, S the set of all non-variables and V the set of all variables occurring in Γ. Let $m = |S|$ and $v = |V|$.
First, in order to compute a subatom B such that $B \in SubAt(\exists r.Z)$, we have a non-deterministic procedure *guess_subatom* that takes a non-variable $\exists r.Z$ and outputs a subatom $B \in SubAt(\gamma(\exists r.Z))$:
Procedure: *guess_subatom*($\exists r.Z$, *length*)

- *Input: A non-variable $\exists r.Z$ and the length of the path.*
- *Output: A subatom $B \in \gamma(\exists r.Z)$ or NULL.*

1. *If length $\ge v \times m^v(1 + 2^{m+1})$, then return NULL.*
2. *If Z is a concept constant, then return $B = \exists r.Z$.*
3. *If Z is a variable, then guess an element $E \in S_1(Z) \cup S_2(Z) \cup \overline{S_3}(Z)$.*
    - *If $E = \exists r'.Y$ then let $B' = guess\_subatom(\exists r'.Y, length + 1)$.*
        - *if $B' = NULL$ then return NULL,*
        - *if $B' \neq NULL$, then return $B = \exists r.B'$.*
    - *If $E \neq \exists r'.Y$, then return $B = \exists r.E$.*

Now we describe our main non-deterministic algorithm to compute a unifier.
Algorithm ($\mathcal{EL}$-top unification procedure)

1. *For each variable X, guess three local sets $S_1(X)$, $S_2(X)$ and $S_3(X)$ such that $S_1(X) \cup S_2(X) \cup \overline{S_3}(X) \neq \emptyset$.*
2. *Check whether there is a variable X occurring in Γ that depends on itself (Definition 10). If it is the case, then return "FAIL". Otherwise, > is a strict dependency order on the variables occurring in Γ.*
3. ***Guessing subatoms**: For each variable X occurring in Γ, we compute the set of subatoms $Sub_\sigma(X)$ as follows.*
    - *Let $S_3(X) = \{\exists r_1.Z_1, ..., \exists r_l.Z_l\}$ and initialize $Sub_\sigma(X) := \emptyset$.*

- *For each i,* $1 \leq i \leq l$*, we compute* $B_i = guess\_subatom(\exists r_i.Z_i, 0)$.
  - *if* $B_i = NULL$ *then return "FAIL",*
  - *if* $B_i \neq NULL$*, then let* $Sub_\sigma(X) := Sub_\sigma(X) \cup \{B_i\}$.

4. ***Computing a substitution*** *: For each variable X occurring in* Γ*, we define* $\sigma(X)$ *as follows.*

   (a) *If X is the least variable w.r.t. the dependency order* $>$*, then* $S_2(X)$ *does not contain any variables. We define*

   $$\sigma(X) = \bigsqcap_{D \in S_1(X) \cup S_2(X)} D \sqcap \bigsqcap_{B_i \in Sub_\sigma(X)} B_i$$

   (b) *We assume that* $\sigma(Y)$ *is defined for all variables* $Y < X$*. Then* $S_2(X)$ *only contains variables Y for which* $\sigma(Y)$ *is defined. Thus we define*

   $$\sigma(X) = \bigsqcap_{A \in S_1(X)} A \sqcap \bigsqcap_{\exists r.Y \in S_2(X)} \exists r.\sigma(Y) \sqcap \bigsqcap_{B_i \in Sub_\sigma(X)} B_i$$

5. *Test whether the substitution* σ *is a unifier of* Γ*. If it is the case, then return* σ*. Otherwise, return "FAIL".*

The algorithm is sound since it returns only unifiers of Γ. Moreover, it always terminates. Thus, to show the correctness of the algorithm, it is enough to show that it is complete.

Lemma 25. *Let* Γ *be a flat* $\mathcal{EL}$*-top unification problem. If* Γ *is solvable, then there is a way of guessing in Step* 1 *subsets* $S_1(X)$*,* $S_2(X)$*,* $S_3(X)$ *and in Step* 4 *subatoms such that the* ***depends on*** *relation is acyclic and the substitution* σ *computed in Step* 4 *is a unifier of* Γ.

Proof. If Γ is solvable, then by Theorem 3, there is a small local unifier γ of Γ. Thus for every variable X occurring in Γ, we have

$$\gamma(X) = A_1 \sqcap ... \sqcap A_n \sqcap \gamma(\exists r_1.X_1) \sqcap ... \sqcap \gamma(\exists r_m.X_m) \sqcap B_{m+1} \sqcap ... \sqcap B_{m+l},$$

where:

- $A_1, ..., A_n$ are concept constants in $Cons(\Gamma)$,
- $\exists r_1.X_1, ..., \exists r_m.X_m, \exists r_{m+1}.X_{m+1}, ..., \exists r_{m+l}.X_{m+l}$ are non-variables of Γ,
- $B_{m+i} \not\sqsupseteq \gamma(D)$, and $B_{m+i} \in SubAt(\gamma(\exists r_{m+i}.X_{m+i}))$, for all $i, 1 \leq i \leq l$ and for each non-variable D of Γ.

Let $Sub_\gamma(X) = \{B_{m+1}, ..., B_{m+l}\}$. We define:

- $S_1(X) = \{A_1, ..., A_n\}$,

- $S_2(X) = \{\exists r_1.X_1, ..., \exists r_m.X_m\}$,
- $S_3(X) = \{\exists r_{m+1}.X_{m+1}, ..., \exists r_{m+l}.X_{m+l}\}$,

First, since γ is a local unifier of Γ, for each variable X occurring in Γ, we have $S_1(X) \cup S_2(X) \cup S_3(X) \neq \emptyset$. Hence Step 1 is successful.
Second, we show that the **depends on** relation defined on the variables occurring in Γ is acyclic. We assume that there is a variable X occurring in Γ that depends on itself. Thus there is a chain $X = Y_0 > Y_1 ... > Y_s = X$, where $s \geq 1$ and $Y_0, ..., Y_s$ are variables occurring in Γ. For each i, $0 \leq i \leq s-1$, if Y_i depends on Y_{i+1} then there is a role name r_{i+1} such that $\exists r_{i+1}.Y_{i+1} \in S_2(Y_i)$. Because of the way $S_2(Y_i)$ is defined, we have $\gamma(Y_i) \sqsubseteq \gamma(\exists r_{i+1}.Y_{i+1}) = \exists r_{i+1}.\gamma(Y_{i+1})$. Thus we have $\gamma(X) \sqsubseteq \exists r_1...r_s.\gamma(X)$ which is impossible since $s \geq 1$. Thus Step 2 is successful.
Third, since γ is a small local unifier of Γ, for each variable X occurring in Γ, $B \in Sub_\gamma(X)$ implies that the role depth B is smaller or equal to $v \times m^v(1 + 2^{m+1})$. Thus B can be computed by **guess_subatom**. Hence $Sub_\sigma(X) = Sub_\gamma(X)$ is computed successfully.
Now we show that the substitution σ computed by the algorithm w.r.t. $S_1(X)$, $S_2(X)$, $Sub_\sigma(X)$ is a unifier of Γ.

- If X is the least variable w.r.t. the dependency order >, then $S_1(X) \cup S_2(X)$ does not contain any variables which implies that $\gamma(D) = D$, for all $D \in S_1(X) \cup S_2(X)$. By definition of $S_1(X)$, $S_2(X)$, $S_3(X)$, $Sub_\sigma(X)$ (Definition 11) and of σ, we have

$$\begin{aligned}\sigma(X) &= \bigsqcap_{1\leq i\leq n} A_i \sqcap \bigsqcap_{1\leq i\leq m} \exists r_i.X_i \sqcap \bigsqcap_{1\leq i\leq l} B_{m+i}\\ &= \bigsqcap_{1\leq i\leq n} A_i \sqcap \bigsqcap_{1\leq i\leq m} \gamma(\exists r_i.X_i) \sqcap \bigsqcap_{1\leq i\leq l} B_{m+i} = \gamma(X)\end{aligned}$$

- We assume that $\sigma(Y) = \gamma(Y)$ holds for all variables $Y < X$. Then $S_2(X)$ contains only variables which are smaller than X. By induction, we have $\sigma(D) = \gamma(D)$, for all $D \in S_2(X)$. By definition of $S_1(X)$, $S_2(X)$, $S_3(X)$, $Sub_\sigma(X)$ (Definition 11) and of σ, we have

$$\begin{aligned}\sigma(X) &= \bigsqcap_{1\leq i\leq n} A_i \sqcap \bigsqcap_{1\leq i\leq m} \sigma(\exists r_i.X_i) \sqcap \bigsqcap_{1\leq i\leq l} B_{m+i}\\ &= \bigsqcap_{1\leq i\leq n} A_i \sqcap \bigsqcap_{1\leq i\leq m} \gamma(\exists r_i.X_i) \sqcap \bigsqcap_{1\leq i\leq l} B_{m+i} = \gamma(X)\end{aligned}$$

□

The following theorem is an immediate consequence of Lemma 3 and Lemma 25.

Theorem 4. *Let Γ be a solvable flat $\mathcal{EL}$-top unification problem. Then the algorithm computes all small local unifiers of* Γ.

4.3 Complexity

In order to justify the complexity of the algorithm, we need to evaluate the size of the substitution computed in Step 4.
First, we define the size of a concept term.

Definition 16. (Size of a concept term)
Let C be a concept term. We define the size $S(C)$ of C is defined as follows:

- *if C is a concept constant, then $S(C) = 1$,*
- *if $C = \exists r.C'$, then $S(C) = 1 + S(C')$,*
- *if $C = C_1 \sqcap C_2$, then $S(C) = S(C_1) + S(C_2)$.*

In the following lemma, we show that the size of a small local unifier has an exponential upper bound.

Lemma 26. *Let Γ be a flat $\mathcal{EL}$-unification problem and γ a small local unifier of Γ. We assume that S is the set of all non-variables of Γ and V is the set of all variables occurring in Γ. Then for every variable X occurring in Γ, we have $S(\gamma(X)) < m^v[3m + 2^m v m^v(1 + 2^{m+1})]$, where $m = |S|$ and $v = |V|$.*

Proof. We prove the lemma in two steps.
Let:

- $S_1(X)$, $S_2(X)$ be local sets of X w.r.t. γ,
- $Sub_\gamma(X) = \{B_1, ..., B_l\}$,
- $N_\gamma(X) = u$, where u is the length of the longest dependency path starting from X,
- $n = |Cons(\Gamma)|$.

First, we show the following:

$$S(\gamma(X)) < m(1 + m^{u-1} + \sum_{i=1}^{u-1} m^i) + [n + 2^m v m^v(1 + 2^{m+1})] \sum_{i=0}^{u-1} m^i \qquad (4.34)$$

We prove (4.34) by induction on the order of X w.r.t. the dependency order $>$.

1. If X is the least variable, then $S_2(X)$ does not contain any variables and $u = 1$. On the other hand, since $S_2(X)$ does not contain any variables, we have

$$S(\gamma(X)) = |S_1(X)| + 2|S_2(X)| + \sum_{i=1}^{l} S(B_i) \qquad (4.35)$$

Moreover, since $S_1(X) \subseteq Cons(\Gamma)$ and $S_2(X) \subseteq S$, we have

$$|S_1(X)| \leq n \text{ and } |S_2(X)| \leq |S| = m. \tag{4.36}$$

By Lemma 24, for all i, $1 \leq i \leq l$, we have

$$S(B_i) < vm^v(1 + 2^{m+1}) \tag{4.37}$$

Since γ is a small local unifier of Γ, we also have

$$l \leq 2^m \tag{4.38}$$

Thus by (4.35), (4.36), (4.37) and (4.38), we have

$S(\gamma(X)) < n + 2m + 2^m vm^v(1 + 2^{m+1})$.

Since $u = 1$, we have

$n + 2m + 2^m vm^v(1 + 2^{m+1}) =$

$m(1 + m^{u-1} + \sum_{i=1}^{u-1} m^i) + [n + 2^m vm^v(1 + 2^{m+1})] \sum_{i=0}^{u-1} m^i$.

Hence (4.34) is satisfied.

2. Assume that for every variable $Y < X$, we have

$$S(\gamma(Y)) < m(1 + m^{n_Y-1} + \sum_{i=1}^{n_Y-1} m^i) + [n + 2^m vm^v(1 + 2^{m+1})] \sum_{i=0}^{n_Y-1} m^i \tag{4.39}$$

where $n_Y = N_\gamma(Y)$.
On the other hand, since $Y < X$, we have:

$$n_Y \leq u - 1 \tag{4.40}$$

We assume that $S_2(X) = \{\exists r_1.Y_1, ..., \exists r_p.Y_p\}$. By (4.36), (4.38), (4.39) and (4.40), we have

$S(\gamma(X)) = |S_1(X)| + \sum_{i=1}^{p}(1 + S(\gamma(Y_i)) + \sum_{i=1}^{l} S(B_i) <$

$n + m[1 + (m.(1 + m^{u-2} + \sum_{i=1}^{u-2} m^i) + (n + 2^m vm^v(1 + 2^{m+1})) \sum_{i=1}^{u-2} m^i] + 2^m vm^v(1 + 2^{m+1}) =$

$m(1 + m^{u-1} + \sum_{i=1}^{u-1} m^i) + [n + 2^m vm^v(1 + 2^{m+1})] \sum_{i=0}^{u-1} m^i$.

Thus (4.34) is satisfied. This completes the proof of (4.34).

Second, since $u \leq v$, we have

$$\sum_{i=0}^{u-1} m^i \leq m^v \tag{4.41}$$

Moreover, we also have

$$m^{u-1} \leq m^{v-1} \leq m^v \text{ and } n \leq m. \tag{4.42}$$

By (4.34), (4.41) and (4.42), we have

$S(\gamma(X)) < m \times (m^v + m^v) + (m + 2^m v m^v (1 + 2^{m+1})) \times m^v =$

$m^v[3m + 2^m v m^v (1 + 2^{m+1})]$. □

Theorem 5. *$\mathcal{EL}$-top unification is in NExpTime.*

Proof. Termination in NExpTime is a consequence of the following facts:

- Guessing three sets $S_1(X)$, $S_2(X)$ and $S_3(X)$ for each variable X occurring in Γ can be done within NExpTime, since:
  - guessing $S_1(X)$, $S_2(X)$ can be done within NP,
  - guessing $S_3(X)$ can be done within NExpTime, because $|S_3(X)|$ is exponential in the size of Γ.
- Computing the **depends on** relation and checking it for acyclicity (Step 2) is polynomial in the size of Γ.
- Computing subatoms for a variable X can de done in NExpTime.
- We now show that checking in Step 4 can be done within ExpTime.
  Let S be the set of all non-variables of Γ and V the set of all variables occurring in Γ. Let $m = |S|$ and $v = |V|$.
  We consider an arbitrary equivalence $C \equiv D \in \Gamma$. Assume that $C = C_1 \sqcap ... \sqcap C_k$, where $C_1, ..., C_k$ are flat atoms of Γ. By Lemma 26, for each i, $1 \leq i \leq k$, we have

  $$S(\gamma(C_i)) < 1 + m^v[3m + 2^m v m^v (1 + 2^{m+1})] \tag{4.43}$$

  By (4.43), we have

  $$S(\gamma(C)) < k(1 + m^v[3m + 2^m v m^v (1 + 2^{m+1})]) \tag{4.44}$$

  It is obvious that

  $$k \leq m + v \tag{4.45}$$

  By (4.44) and (4.45), we have

  $$S(\gamma(C)) < (m + v)(1 + m^v[3m + 2^m v m^v (1 + 2^{m+1})]) \tag{4.46}$$

This inequality also holds for $S(\gamma(D))$. Thus size of each equivalence in Γ under γ is of at most exponential in the size of Γ and subsumption checking in the algorithm can be done within exponential time in the size of Γ. Overall, checking in Step 4 can be done within ExpTime.

$\square$

Chapter 5

$\mathcal{EL}$-top unification is PSPACE-hard

In this chapter, we show that $\mathcal{EL}$-top unification is PSPACE-hard. We do this by reducing the Finite State Automata Intersection problem which has been shown to be PSPACE-complete by Kozen [8]. This result is interesting because though $\mathcal{EL}$-top is less expressive than $\mathcal{EL}$, $\mathcal{EL}$-top unification is more difficult than it is for $\mathcal{EL}$ which has been shown to be NP-complete [4].

5.1 Finite State Automata Intersection problem

We first introduce the general notion of a non-deterministic finite state automaton, and then we define a deterministic finite state automaton as a special case of the non-deterministic one.

Definition 17. (A non-deterministic finite state automaton)
A non-deterministic finite state automaton $A = \{Q, \Sigma, I, \Delta, F\}$ consists of

- *a finite set of states Q,*
- *a finite alphabet Σ,*
- *a set of initial states $I \subseteq Q$,*
- *a set of final states $F \subseteq Q$.*

A **path** in the automaton is a sequence $q_0 a_1 q_1 a_2 ... a_n q_n$, where $(q_{i-1}, a_i, q_i) \in \Delta$ for $1 \leq i \leq n$. We will often abbreviate such a path as $q_0 \xrightarrow{a_1 ... a_n}_A q_n$. The path is **successful** if $q_0 \in I$ and $q_n \in F$.
We say that ω is accepted by A iff there is a successful path $q_0 \xrightarrow{\omega}_A q_n$ on A. We will use the notion of a word accepted by an automaton A starting from a state q,

i.e., ω is accepted by $A(q)$ iff there is a path $q \xrightarrow{\omega}_A q'$ on A such that $q' \in F$.
The automaton A accepts the following language:

$$L(A) = \{\omega \in \Sigma^* \mid q_0 \xrightarrow{\omega}_A q_n \text{ is a successful path in } A\}. \tag{5.1}$$

A deterministic finite state (DFS) automaton is a non-deterministic finite state automaton with additional properties. It is defined formally as follows.

Definition 18. (A deterministic finite state automaton)
*An automaton $A = \{Q, \Sigma, I, \Delta, F\}$ is called **deterministic** iff:*

- $|I| = 1$*, i.e.,* $I = \{q_0\}$*,*
- Δ *is functional, i.e., for every* $q \in Q$ *and every* $a \in \Sigma$ *there is exactly one* $q' \in Q$ *such that* $(q, a, q') \in \Delta$*.*

We can assume that for every $q \in Q$, there is a successful path on A starting from q. Otherwise, q can be safely removed from Q. Moreover, we can also assume that A has exactly one final state. Since we are going to deal with DFS automata, from now on we define A as a tuple $\{Q, \Sigma, \Delta, q_0, q_f\}$, where q_0 is the initial state and q_f is the final state.
The Finite State Automata Intersection problem is defined as follows.

Definition 19. (The Finite State Automata Intersection problem)
Let $A_1, ..., A_n$ be DFS automata, where $A_i = \{Q^i, \Sigma, \Delta^i, q^i_o, q^i_f\}$. Decide whether the set $L(A_1) \cap \cdots \cap L(A_n)$ is empty or not.

5.2 Reduction of the Finite State Automata Intersection problem to $\mathcal{EL}$-top unification problem

In order to reduce a given Finite State Automata Intersection problem to $\mathcal{EL}$-top unification problem Γ, we first define the signature of Γ. Let $A_1, ..., A_n$ be DFS automata.
The sets N_c, N_v and N_r are defined as follows:

- $N_c = \{A\}$,
- $N_v = \{X^i_j \mid q^i_j \in Q^i, 1 \leq i \leq n\} \cup \{Y\}$,
- $N_r = \{a \mid a \in \Sigma\}$

Second, the set of subsumptions Γ_i for each i, $1 \leq i \leq n$ is defined as follows:

1. Initialize $\Gamma_i := \emptyset$,
2. For each $q^i_j \in Q^i$:
    - If $q^i_j = q^i_f$, then $\Gamma_i := \Gamma_i \cup \{A \sqcap \bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k \sqsubseteq X^i_j\}$.

- If $q^i_j \neq q^i_f$, then $\Gamma_i := \Gamma_i \cup \{\bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k \sqsubseteq X^i_j\}$.

3. Let $\Gamma_i := \Gamma_i \cup \{X^i_0 \sqsubseteq Y\}$.

Third, let $\Gamma = \bigcup_{1 \leq i \leq n} \Gamma_i$.
Since $C \sqsubseteq D$ is equivalent to $C \sqcap D \equiv C$, we can consider Γ as a flat $\mathcal{EL}$-top unification problem.
In the remaining part of this section, we prove that Γ defined in this way is unifiable in $\mathcal{EL}$-top iff the intersection of languages $L(A_1), \ldots, L(A_n)$ is not empty. First, we prove the following lemma.

Lemma 27. *If γ is a small local unifier of Γ, then for all $q^i_j \in Q^i$, $1 \leq i \leq n$, $\exists \omega.A \in SubAt(\gamma(X^i_j))$ implies that ω is accepted by $A_i(q^i_j)$, where $\omega \in \Sigma^*$.*

Proof. We prove this lemma by induction on the length of ω.

1. If $|\omega| = 0$, then $A \in SubAt(\gamma(X^i_j))$. If $q^i_j = q^i_f$, then obviously $\omega = \epsilon$ is accepted by $A_i(q^i_j)$. Otherwise, we have $\bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k \sqsubseteq X^i_j \in \Gamma$, and thus by Lemma 4, $A \in SubAt(\gamma(\bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k))$ which is impossible.

2. We assume that $\omega = v\omega'$, where $|\omega'| \geq 0$ and $v \in \Sigma$. Consider the following cases:

   - If $q^i_j = q^i_f$, then we have $A \sqcap \bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k \sqsubseteq X^i_j$. By Lemma 4, $\exists \omega.A \in SubAt(\gamma(X^i_j))$ implies that $\exists \omega.A \in SubAt(A \sqcap \gamma(\bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k))$. Since $|\omega| > 0$, $\exists \omega.A \neq A$. Thus there is a transition $(q^i_j, a_t, q^i_k) \in \Delta^i$ such that $v = a_t$ and $\exists \omega.A \in SubAt(\gamma(\exists a_t.X^i_k))$. Hence $\omega' \in SubAt(\gamma(X^i_k))$. By induction, ω' is accepted by $A_i(q^i_k)$ and thus ω is accepted by $A_i(q^i_j)$.
   - If $q^i_j \neq q^i_f$, then we have $\bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k \sqsubseteq X^i_j$. By Lemma 4, $\exists \omega.A \in SubAt(\gamma(X^i_j))$ implies that $\exists \omega.A \in SubAt(\gamma(\bigsqcap_{(q^i_j, a_t, q^i_k) \in \Delta^i} \exists a_t.X^i_k)$. Thus there is a transition $(q^i_j, a_t, q^i_k) \in \Delta^i$ such that $v = a_t$ and $\exists \omega.A \in SubAt(\gamma(\exists a_t.X^i_k))$. Hence, we have $\exists \omega'.A \in SubAt(\gamma(X^i_k))$. By induction, ω' is accepted by $A_i(q^i_k)$ and thus ω is accepted by $A_i(q^i_j)$.

□

In the next lemma, we show that if the corresponding $\mathcal{EL}$-top unification problem is solvable, then the intersection of DFS automata is not empty.

Lemma 28. *If Γ is solvable, then $L(A_1) \cap ... \cap L(A_n) \neq \emptyset$.*

Proof. If Γ is solvable, then by Theorem 3, it has a small local unifier γ. Let $B \in SubAt(\gamma(Y))$. Since γ is a small local unifier of Γ and $Cons(\Gamma) = \{A\}$, there is a word $\omega \in \Sigma^*$ such that $B = \exists \omega.A$.

For each i, $1 \leq i \leq n$, since $\gamma(X_0^i) \sqsubseteq \gamma(Y)$, by Lemma 4, $B \in SubAt(\gamma(Y))$ implies that

$$B \in SubAt(\gamma(X_0^i)) \tag{5.2}$$

On the other hand, by Lemma 27, (5.2) yields that ω is accepted by $A_i(q_0^i)$, i.e., $\omega \in L(A_i)$.
Thus $\omega \in \bigcap_{1\leq i\leq n} L(A_i)$ which implies that $L(A_1) \cap ... \cap L(A_n) \neq \emptyset$. □

The following lemma shows that the non-empty intersection of DFS automata implies the solvability of the corresponding $\mathcal{EL}$-top unification problem.

Lemma 29. *If $L(A_1) \cap ... \cap L(A_n) \neq \emptyset$, then Γ is solvable.*

Proof. Let $\omega \in L(A_1) \cap ... \cap L(A_n)$ and $\omega = b_1 \dots b_m$, where $\{b_1, ..., b_m\} \subseteq \Sigma$. Since $\omega \in L(A_i)$ for each i, $1 \leq i \leq n$, there is a successful path $q_0 \xrightarrow{b_1...b_m}_{A_i} q_m$. Thus we have $q_0 = q_0^i$ and $q_m = q_f^i$. For each $q_j^i \in Q^i$, let ω_j^i be a minimal word w.r.t. the length, accepted by $A_i(q_j^i)$. We define the substitution set $S(X_j^i)$ for each $X_j^i \in N_v$ as follows:

- We initialize $S(X_j^i) := \{\exists\omega_j^i.A\}$,
- For each $q_j^i \in Q^i$ and for each $(q_j^i, a_t, q_k^i) \in \Delta^i$:
  - If $q_j^i = q_p$ and $a_t = b_{p+1}$ for some $0 \leq p \leq m-1$, then let $At_{j,t}^i = \exists b_{p+1}...b_m.A$ and $S(X_j^i) := S(X_j^i) \cup \{At_{j,t}^i\}$.
  - If $q_j^i \neq q_p$ or $a_t \neq b_{p+1}$ for all p, $0 \leq p \leq m-1$, then let $At_{j,t}^i = \exists a_t\omega_k^i.A$ and $S(X_j^i) := S(X_j^i) \cup \{At_{j,t}^i\}$.

Now we define the substitution γ:

1. $\gamma(X_j^i) = \bigsqcap_{B\in S(X_j^i)} B$,
2. $\gamma(Y) = \exists\omega.A$.

We show that γ is a unifier of Γ. Without loss of generality, it is enough to prove that γ satisfies all the subsumptions in Γ_i for an arbitrary i, $1 \leq i \leq n$.
Notice that the following is obviously true:

$$\bigsqcap_{(q_j^i,a_s,q_k^i)\in\Delta^i} \gamma(\exists a_s.X_k^i) \sqsubseteq \gamma(\exists a_t.X_k^i), \text{ for all } (q_j^i, a_t, q_k^i) \in \Delta_i. \tag{5.3}$$

By the definitions of Γ and γ, we need to show that for all $X_j^i \in N_v$, γ satisfies:

$$A \sqcap \bigsqcap_{(q_j^i,a_s,q_k^i)\in\Delta^i} \exists a_s.X_k^i \sqsubseteq X_j^i, \quad \text{if } q_j^i = q_f^i. \tag{5.4}$$

and

$$\bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \exists a_s.X^i_k \sqsubseteq X^i_j, \quad \text{if } q^i_j \neq q^i_f. \tag{5.5}$$

Since $\gamma(X^i_j) = \bigsqcap_{B\in S(X^i_j)} B$, by Corollary 1, it is enough to show that for all $B \in S(X^i_j)$, γ satisfies

$$A \sqcap \bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \exists a_s.X^i_k \sqsubseteq B, \quad \text{if } q^i_j = q^i_f. \tag{5.6}$$

and

$$\bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \exists a_s.X^i_k \sqsubseteq B, \quad \text{if } q^i_j \neq q^i_f. \tag{5.7}$$

By the definition of $S(X^i_j)$, we have:

$$B = \omega^i_j \quad \text{or} \quad B = At^i_{j,t}. \tag{5.8}$$

First, we show that

$$\bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \gamma(\exists a_s.X^i_k) \sqsubseteq At^i_{j,t} \tag{5.9}$$

We consider the following cases:

- If $q^i_j = q_p$ and $a_t = b_{p+1}$ for some $0 \leq p \leq m-1$, then we have $At^i_{j,t} = \exists b_{p+1}...b_m.A$. On the other hand, we have $q^i_k = q_{p+1}$ which implies that $\exists b_{p+2}...b_m.A \in S(X^i_k)$, where $\exists b_{p+2}...b_m.A = A$ if $p+2 > m$. Thus $\gamma(\exists a_t.X^i_k) \sqsubseteq At^i_{j,t}$. By (5.3), we have $\bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \gamma(\exists a_s.X^i_k) \sqsubseteq At^i_{j,t}$.

- If $q^i_j \neq q_p$ or $a_t \neq b_{p+1}$ for all p, $0 \leq p \leq m-1$, then we have $At^i_{j,t} = \exists a_t\omega^i_k.A$. On the other hand, $\exists\omega^i_k.A \in S(X^i_k)$ which implies that

  $$\gamma(X^i_k) \sqsubseteq \exists\omega^i_k.A \tag{5.10}$$

  By (5.3) and (5.10), we have $\bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \gamma(\exists a_s.X^i_k) \sqsubseteq \exists a_t\omega^i_k.A = At^i_{j,t}$

This completes the proof of (5.9). By (5.9), we have:

$$A \sqcap \bigsqcap_{(q^i_j,a_s,q^i_k)\in\Delta^i} \gamma(\exists a_s.X^i_k) \sqsubseteq At^i_{j,t} \tag{5.11}$$

Second, we prove that (5.6) and (5.7) are satisfied by γ. For each $q^i_j \in Q^i$, we consider the following cases:

1. $q^i_j = q^i_f$. In this case, we have to show that (5.6) is satisfied by γ.
   Since $q^i_j = q_m = q^i_f$, we have $\exists\omega^i_j.A = A \in S(X^i_j)$. It is obvious that

$$A \sqcap \bigsqcap_{(q^i_j, a_s, q^i_k) \in \Delta^i} \gamma(\exists a_s.X^i_k) \sqsubseteq A = \exists\omega^i_j.A \tag{5.12}$$

   By (5.8), (5.12) and (5.11), γ satisfies (5.6).

2. $q^i_j \neq q^i_f$. In this case, we have to show that (5.7) is satisfied by γ.
   It is obvious that there is a transition $(q^i_j, a_t, q^i_k) \in \Delta^i$ such that $\omega^i_j = a_t\omega^i_k$. On the other hand, since $\exists\omega^i_k.A \in S(X^i_k)$, we have

$$\gamma(X^i_k) \sqsubseteq \exists\omega^i_k.A \tag{5.13}$$

   Thus by (5.3) and (5.13), we have

$$\bigsqcap_{(q^i_j, a_s, q^i_k) \in \Delta^i} \gamma(\exists a_s.X^i_k) \sqsubseteq \gamma(\exists a_t.X^i_k) \sqsubseteq \exists a_t\omega^i_k.A = \exists\omega^i_j.A \tag{5.14}$$

   By (5.8), (5.14) and (5.9), γ satisfies (5.7).

We have shown that the subsumptions (5.6) and (5.7) are satisfied by γ. Third, we show that γ satisfies the subsumption $X^i_0 \sqsubseteq Y$. Since $q^i_0 = q_0$, we have:

1. If $\omega = \epsilon$, then $q^i_0 = q_m$. Thus we have $\omega^i_0 = \epsilon$. By the definition of $S(X^i_0)$, $\exists\omega^i_0.A = A \in S(X^i_0)$. Hence $\exists\omega.A = A \in S(X^i_0)$.

2. If $\omega \neq \epsilon$, then $q^i_0 \neq q_m$ and thus $m \geq 1$. Moreover, since $\omega \in L(A_i)$, there is a transition $(q^i_0, a_t, q_1) \in \Delta^i$. Hence we have $At^i_{0,t} = \exists\omega.A$. By the definition of $S(X^i_0)$, we have $At^i_{0,t} \in S(X^i_0)$ and thus $\exists\omega.A \in S(X^i_0)$.

In all cases, we have $\exists\omega.A \in S(X^i_0)$. Thus $\gamma(X^i_0) \sqsubseteq \exists\omega.A = \gamma(Y)$.
We have shown that Γ_i is satisfied by γ, for all i, $1 \leq i \leq n$. □

The following theorem is a consequence of Lemma 28 and Lemma 29.

Theorem 6. *$L(A_1) \cap ... \cap L(A_n) \neq \emptyset$ iff Γ is solvable.*

PSPACE-hardness property of $\mathcal{EL}$-top unification follows immediately from Theorem 6 and the fact that the Finite State Automata Intersection problem is PSPACE-complete.

Chapter 6

Conclusion

In this thesis, we have shown that $\mathcal{EL}$-top unification problem is decidable.
In Chapter 3, we have introduced the notion of local unifiers and proved that $\mathcal{EL}$-top unification has local property, i.e, a local unifier can be constructed from elements in the goal. However, in the end of this chapter, we have shown, by an example, that the locality is not strong enough to help us to obtain a decision procedure for $\mathcal{EL}$-top unification. In Chapter 4, we restricted further the set of local unifiers and introduced the notion of small local unifiers so that we were able to construct an NExpTime decision procedure for $\mathcal{EL}$-top unification. Interestingly, in Chapter 5, we have shown that $\mathcal{EL}$-top unification is PSPACE-hard by reducing Finite State Automata Intersection problem to $\mathcal{EL}$-top unification. By this result, even though $\mathcal{EL}$-top is less expressive than $\mathcal{EL}$, $\mathcal{EL}$-top unification is in fact more difficult than it is for $\mathcal{EL}$, since $\mathcal{EL}$-unification has been shown to be NP-complete. Unless PSPACE=NP, there is no algorithm for $\mathcal{EL}$-top unification in NP complexity class.
However, the exact complexity of the problem has not been discovered yet. The NExpTime decision procedure described in Chapter 4 can certainly be improved. In future, we will consider how to obtain another procedure with smaller complexity, e.g, an ExpTime procedure. Furthermore, we also want to obtain a better lower bound of the complexity by showing ExpTime-hardness.

Bibliography

[1] Franz Baader. *Unification in commutative theories*, J. of Symbolic Computation, 8(5):479-497, 1989.

[2] Franz Baader, Diego Calvanese, Deborah McGuinness, Deniele Nardi, and Peter F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications.* Cambridge University Press, 2003.

[3] Franz Baader and Ralf Küsters. *Matching in description logics with existential restrictions*. In Proc. of the 7th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR 2000), pages 261-272, 2000.

[4] Franz Baader and Barbara Morawska. *Unification in the description logic $\mathcal{EL}$.* Proceedings of the 20th International Conference on Rewriting Technique and Applications (RTA 2009), Lectures Notes in Computer Science. Spring-Verlag, 2009.

[5] Franz Baader and Paliath Narendran. *Unification of concepts terms in description logics*, J. of Symbolic Computation, 31(3):277-305, 2001.

[6] Franz Baader and Wayne Snyder. Unification theory. In J.A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, volume I, pages 447-533. Elsevier Science Publishers, 2001. Springer-Verlag, 2001.

[7] Gene Ontology. *http://www.geneontology.org/.*

[8] Dexter Kozen. *Lower bounds for natural proof systems*. Proc. 18th Ann. Symp. on Foundations of Computer Science, IEEE Computer Society, Long Beach, CA, 254-266, October 1977.

[9] R. Küsters. *Non-standard Inferences in Description Logics*, Lecture Notes in Artificial Intelligence 2100. Springer-Verlag, 2001.

[10] David McAllester. *Automatic Recognition of Tractability in Inference Relations.* JACM, Vol. 40(2), 1993.

[11] OWL2 standard. *http://www.w3.org/TR/owl2-profiles/.*

[12] Alan Rector and Ian Horrocks. *Experience building a large, re-usable medical ontology using a description logic with transitivity and concept inclusions*. In Proceedings of the Workshop on Ontological Engineering, AAAI Spring Symposium (AAAI'97), Stanford, AAAI Press, 1997.

[13] SNOMED Ontology. *http://www.ihtsdo.org/snomed-ct/*.

[14] Viorica Sofronie-Stokkermans. *Locality and subsumption testing in $\mathcal{EL}$ and some of its extension*. Proceedings of AiML, 2008.