Quantifier Elimination in Quantified Propositional Łukasiewicz Logic

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Kurzfassung

In dieser Masterarbeit hat der Verfasser die minimale Erweiterung der quantifiziert propositionalen Łukasiewicz Logik bestimmt, die Quantorenelimination zulässt. Um dieses Ergebnis zu erzielen wurden folgendes Aufgaben gelöst: Wir haben die Łukasiewicz Aussagenlogik durch allen Divsionsoperatoren erweitert und haben die Vollständigkeit der erweiterten Logik bewiesen. Dann haben wir unter Verwendung des McNaughton Theorems die sogenannten Mini max Punkte der Wahrheitfunktionen der Łukasiewicz Logik ermittelt und diese Punkte in der Sprache der durch alle Divisionsoperatoren erweiterte Łukasiewicz Logik ausgedrückt. Die gewnschte Zulsäsigkeit der Quantorelimination folgt durch Kombination der oben genannten Ergebnisse. Als logische Folgerungen ergibt sich dass die mit allen Divisionsoperatoren erweiterte quantifiziert propositionale Łukasiewicz Logik semantisch Vollständig, entscheidbar und daher rekursiv aufzählbar ist.

Abstract

In this master thesis we have determined the minimal extension of quantified propositional Łukasiewicz logic which admits quantifier elimination. To obtained this result we solved following problems: we have extended Łukasiewicz logic by all division operators and have proven completeness of the extended logic. Afterwards we used McNaughton theorem to determine so called minimax points of Łukasiewicz logic truth functions. We expressed this points in the language of Łukasiewicz logic extended by all division operators, then by combining the above mentioned results we have obtained the desired admissibility quantifier elimination. As a corollary we obtained that quantified propositional Łukasiewicz logic extended by all division operators is semantically complete, decidable and hence recursively enumerable.

Introduction

Corresponding to the thesis topic we will work with quantified propositional Łukasiewicz logic and we will discuss conditions for the elimination of propositional quantifiers in it. Propositional quantifiers are defined in a natural way by suprema and infima.
Quantified propositional logic is a natural closure of propositional logic. Unlike in classical logic, adding propositional quantifiers to Fuzzy logics in many cases increases the expressive power of the logic. Propositional quantifiers allows to express complicated properties more naturally. For example satisfiability and validity of formulas are easily expressible within the extended logic by such quantifiers. Using propositional quantifiers we can construct left and right interpolants.
A Logic admits quantifier elimination if for each formula F we can find formula F' such that we can prove equivalence $\mathrm{F} \longleftrightarrow \mathrm{F}'$ in this Logic and F' is quantifier free and contains only free variables of F.
As already mentioned we will concentrate on quantified propositional Łukasiewicz Logic and quantifier elimination in this logic. It is already known that quantified propositional Łukasiewicz Logic does not admit quantifier elimination in the original language.
In this paper we will provide the minimal extension of quantified propositional Łukasiewicz Logic which admits quantifier elimination. Minimality is meant in the following sense: there is no other extension with smaller (w.r.t $\subseteq$) signature
Admissibility of quantifier elimination is related to the interpolation property, more precisely, quantifier elimination implies uniform interpolation and consequently the lack of uniform interpolation implies non-existence of quantifier elimination and more generally without interpolation we have no quantifier elimination. From this and from the fact that Propositional Łukasiewicz Logic extended by a finite number of division operators does not admit interpolation property we were motivated to check whether Łukasiewicz Logic extended by all division operators (a.k.a root operators) admits quan-

tifier elimination or not, answer is positive.
Root operators $\Box_n y$ are defined as a maximal solution of equation $\underbrace{x\&...\&x}_{n} = y$. In Łukasiewicz Logic root operator and divisions on natural numbers are similar and mutually expressible. Therefore we will use the name *division operator* as we are working with Łukasiewicz Logic . The extension by all division operators is a very natural extension for Łukasiewicz Logic. Division operators provide us an opportunity to express rational numbers in $[0, 1]$, and their product, to construct interpolants and finally to eliminate quantifiers.
It has already been proven that quantified propositional Gödel logic on [0,1] admits quantifier elimination.(This is also known for following truth value sets $V_\downarrow = \{0\} \cup \{\frac{1}{n} : n \geq 1\}$ and $V_\uparrow = \{1\} \cup \{1 - \frac{1}{n} : n \geq 1\}$.)Our method completely differs from the method used in the proofs of the quantifier elimination properties mentioned: Our method is semantical and uses **McNaughton Theorem** which states that Łukasiewicz Logic truth functions are piecewise linear functions and using analytical properties of piecewise linear functions we are constructing tautologically equivalent quantifier free formula for all quantified formulas.
To complete the proof of quantifier elimination we will need the completeness of Łukasiewicz Logic extended by all division operators. to achieve this aim we will algebraize this logic and prove completeness in the same manner as for propositional Łukasiewicz Logic in [1].

In the first chapter we will provide some basic definitions and facts about t-norms and algebras which we will need in the following chapters. In the second chapter we will extend basic many valued logic by all division operators and will provide axiomatization of it. In the third chapter we will give proof of completeness of basic many valued logic extended by all division operators and next to it in subsequent chapter we will prove completeness of Łukasiewicz Logic extended with all division operators. In the fifth chapter we will discuss quantifier elimination, and finally we will give proof of the admissibility quantifier elimination and some corollaries of this theorem.

Chapter 1

Basic Definitions and Facts

We are working with many valued propositional Logics. We will take unit interval [0,1] for our set of truth values and will work with a logical calculi in which each connective C^n has a truth function $f_c : [0,1]^n \longrightarrow [0,1]$ determining for any n-tuple of formulas $(\phi_1, \ldots \phi_n)$ truth degree of the formula $C^n(\phi_1, \ldots \phi_n)$ from the truth degrees of $\phi_1, \ldots \phi_n$. Since any many valued logic should be a generalization of classical, for 0 and 1 our truth functions of well known connectives should behave classically. In the first section we will define the truth function for conjunction, and then we will determine whole propositional calculus. To prove completeness of our propositional calculus we need to algebraize it, for this we need some basic definitions and facts from algebra and this will be in the second section. In third section we will provide some facts about ordered Abelian groups which we will need for the proof of completeness of Łukasiewicz Logic extended by all division operators.

1.1 t-norms, their residua and divisibility witnesses

Intuitive understanding of conjunction is as follows: a big truth degree of $\phi \& \psi$ should indicate that both the truth degree of ϕ and truth degree of ψ is big, without any preference between ϕ and ψ . Thus it is natural to assume that truth function of conjunction is non-decreasing in both arguments, 1 is its unit element and 0 its zero element. These requirements ar met by the following definition:

Definition 1.1.1. (t-norm) Binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is t-norm if

(i) $*$ is commutative and associative i.e. for all $x, y, z \in [0,1]$ holds following equations

$$x * y = y * x$$
$$x * (y * z) = (x * y) * z$$

(ii) $*$ is non decreasing in both arguments, i.e.

$$x_1 \leqslant x_2 \text{ implies } x_1 * y \leqslant x_2 * y$$
$$y_1 \leqslant y_2 \text{ implies } x * y_1 \leqslant x * y_2$$

(iii) For all $x \in [0,1]$, $1 * x = x$ and $0 * x = 0$

Example 1.1.2. The following are most important examples of continuous($*$ is continuous means that it is continuous mapping from $[0,1]^2$ to $[0,1]$) t-norms:
1. Łukasiewicz t-norm: $x * y = max(0, x + y - 1)$
2. Gödel t-norm: $x * y = min(x, y)$
3. Product t-norm $x * y = x \cdot y$

Let us consider now implication. If we will summarize all intuitive requirements we will come to following:

Definition 1.1.3. (**residua**) If $*$ is t-norm then the operation $x \Rightarrow y = max\{z | x * z \leqslant y\}$ is called *residuum* of t-norm $*$.

Let us write some basic Lemmas which follows immediately from definition of residuum.

Lemma 1.1.4. *(**Residuation**) Let $*$ be continuous t-norm and $\Rightarrow$ residuum of it, then following holds*

$$z \leqslant (x \Rightarrow y) \textit{ iff } x * z \leqslant y$$

Lemma 1.1.5. *For each continuous t-norm and its residuum $\Rightarrow$ following holds*

$$\textit{(i) } x \leqslant y \textit{ iff } (x \Rightarrow y) = 1$$

$$\textit{(ii) } (1 \Rightarrow x) = x$$

Theorem 1.1.6. *The following operations are residua of three t-norms of 1.1.2: $x \Rightarrow y = 1$ and $x \leqslant y$ and for $x > y$*
(i) Lukasiewicz implication: $x \Rightarrow y = 1 - x + y$
(ii) Gödel implication: $x \Rightarrow y = y$
(iii) Goguen Implication: $x \Rightarrow y = \frac{y}{x}$ (residuum of product conjunction)
◀ *assume $x > y$*
(i) $x * z = y$ iff $x + z - 1 = y$ iff $z = 1 - x - y$, thus $z = max\{z|x * z \leqslant y\}$
(ii) $x * z = y$ iff $min(x, z) = y$ iff $z = y$
(iii)$x * z = y$ iff $x \cdot z = y$ iff y/x ▶

Remark 1.1.7. If we have a continuous t-norm then we can define min and max functions.
Let us define new operations $\cap$ and $\cup$.
$x \cap y = x * (x \Rightarrow y)$
$x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$
and show that $x \cap y = min(x, y)$ and $x \cup y = max(x, y)$
(i) ($\cap = min$) If $x \leqslant y$ then $x \Rightarrow y = 1$ and $x * (x \Rightarrow y) = x * 1 = x = min(x, y)$. If $x > y$ then $x * (x \Rightarrow y) = x * max\{z|x * z \leqslant y\} = max\{x * z|x * z \leqslant y\} = y = min(x, y)$. Here we used that $f(z) = x * z$ is continuous function s.t. $f(0) = 0$ and $f(1) = x > y$ and because of this there is z such that $f(z) = x * z = y$.
(ii)($\cup = max$) If $x \leqslant y$ then $(x \Rightarrow y) \Rightarrow y = 1 \Rightarrow y = y$. Since by residuation $y * (y \Rightarrow x) \leqslant x$ is true, $y \leqslant (y \Rightarrow x) \Rightarrow x$ holds. So we have $x \cup y = y = max(x, y)$, The case $y \leqslant y$ is completely symmetric.

Now we will define next important operator for us. This is the division operator or in the other words the root operator. But before we will define our new operators, for correctness of our definition we need following:

Lemma 1.1.8. *For each $x \in [0, 1]$ and for each $n \in \{1, 2, 3 \ldots\}$ if $*$ is a continuous t-norm then there exists $y \in [0, 1]$ such that $\underbrace{y * \ldots * y}_{n\ times} = y^n = x$*

◀ *If $x = 0$ then $y = 0$. If $x = 1$ then $y = 1$. Now if $0 < x < 1$, consider function $f(y) = y^n$, because of continuity of $*$, f is also continuous. $f(0) = 0, f(1) = 1$, since f is also continuous, for all $x \in (f(0), f(1)) = (0, 1)$ there exists $y \in [0, 1]$ s.t. $f(y) = x$.* ▶

Definition 1.1.9. (**divisibility witnesses**) Assume $*$ is a continuous t-norm. For all $i \in \{1, 2, 3 \ldots\}$ define *divisibility witness* as a maximal solution of equation $y^i = x$ and denote by $\blacksquare_i x$, so $\blacksquare_i x = max\{y|y^i = x\}$. We will call $\blacksquare_i$ *division operator* or *root operator.*

Remark 1.1.10. $\blacksquare_i$ has different names, in Product Logic it coincides with root operator. In Łukasziewic Logic it is similar to division.

We will summarize the basic properties of $\blacksquare_i$ in following:

Lemma 1.1.11. *If $*$ is continuous t-norm and $\blacksquare_i$ is its root operator then following holds for all $i \in \{1, 2, 3 \ldots\}$ and for all $x, y \in [0, 1]$:*
(i) $(\blacksquare_i x)^i = x$
(ii) $x \leqslant \blacksquare_i(x^i)$
(iii) If $x < y$ *then* $\blacksquare_i x \leqslant \blacksquare_i y$
(iv) $\blacksquare_i x * \blacksquare_i y \leqslant \blacksquare_i(x * y)$
(v) $\blacksquare_i(\blacksquare_j x) = \blacksquare_{i\cdot j} x$
(vi) $x \leqslant \blacksquare_i x$
◀
(i) and (ii) are Obvious
(iii) Assume $x < y$. It is clear that if $a \leqslant b$ then $a^i \leqslant b^i$, so if $\blacksquare_i x > \blacksquare_i y$ holds then we would have $x = (\blacksquare_i x)^i \geq (\blacksquare_i y)^i = y$ and it is contradiction, so $\blacksquare_i x \leqslant \blacksquare_i y$ is true.
(iv) Because of commutativity and associativity of $*$ and (i) we have $(\blacksquare_i x * \blacksquare_i y)^i = x * y$, and it means that $\blacksquare_i x * \blacksquare_i y \leqslant \blacksquare_i(x * y)$ is true.
(v) Assume $\blacksquare_j x = z$ and $\blacksquare_i z = y$, then $y^{i\cdot j} = z^j = x$, so $\blacksquare_i(\blacksquare_j x) \leqslant \blacksquare_{i\cdot j} x$. $(\blacksquare_{i\cdot j} x)^{i\cdot j} = x$ iff $((\blacksquare_{i\cdot j} x)^i)^j = x$ so $(\blacksquare_{i\cdot j} x)^i \leqslant z$ and by (i) and (iii) it means that $\blacksquare_{i\cdot j} x \leqslant y = \blacksquare_i(\blacksquare_j x)$
(vi) It is clear that for all $x, y \in [0, 1]$ $x * y \leqslant x$. By (i) we have $(\blacksquare_i x)^i = x$ if $i > 1$ we can represent $(\blacksquare_i x)^i$ as $(\blacksquare_i x * (\blacksquare_i x)^{i-1})$. Now rewrite equation from (i) $(\blacksquare_i x * (\blacksquare_i x)^{i-1}) = x$, but $\blacksquare_i x \geq \blacksquare_i x * (\blacksquare_i x)^{i-1} = x$. Case $i = 1$ is trivial.
▶

Theorem 1.1.12. *For all $i \in \{1, 2, 3 \ldots\}$ and for all $x, y \in [0, 1]$ $y * (y^i \Rightarrow x) \leqslant \blacksquare_i x$*
◀ *At first we need to prove that for all $i \in \{1, 2, 3 \ldots\}$ and for all $a, b \in [0, 1]$ $a \Rightarrow b \leqslant \blacksquare_i a \Rightarrow \blacksquare_i b$. Assume $a \Rightarrow b = t$, then by residuation $a * t \leqslant b$. From (iii) of Lemma 2.1.10 we have $\blacksquare_i(a * t) \leqslant \blacksquare_i b$, again by Lemma 2.1.10 but now from (iv) we have $\blacksquare_i a * \blacksquare_i t \leqslant \blacksquare_i b$, and by residuation it means that $\blacksquare_i t \leqslant \blacksquare_i a \Rightarrow \blacksquare_i b$, but from this and (ii) of Lemma 2.1.10 we have $a \Rightarrow b = t \leqslant \blacksquare_i t \leqslant \blacksquare_i a \Rightarrow \blacksquare_i b$, so $a \Rightarrow b \leqslant \blacksquare_i a \Rightarrow \blacksquare_i b$ is proven, from this by residuaion we have following $\blacksquare_i a * (a \Rightarrow b) \leqslant \blacksquare_i b$. Take now for b*

-x and for a - y^i, So following holds $\blacksquare_i(y^i) * (y^i \Rightarrow x) \leqslant \blacksquare_i x$. From (ii) of Lemma 2.1.10 and from the fact that $*$ is non-decreasing in both arguments we have following : $y * (y^i \Rightarrow x) \leqslant \blacksquare_i(y^i) * (y^i \Rightarrow x) \leqslant \blacksquare_i x$. So we proved that for all $i \in \{1, 2, 3 \ldots\}$ and for all $x, y \in [0, 1]$ $y * (y^i \Rightarrow x) \leqslant \blacksquare_i x$ ▶

Theorem 1.1.13. *The following operations are divisibility witnesses of three t-norms of 1.1.2*
(i) Łukasiewicz divisibility witness or division operator: $\blacksquare_i x = \frac{x}{i} + \frac{i-1}{i}$
(ii) Gödel divisibility witness: $\blacksquare_i x = x$
(iii) Goguen divisibility witness or root operator: $\blacksquare_i x = x^{\frac{1}{i}}$
◀
(i) $x = y^i = ((...(y + y - 1) + y - 1) + y - 1)... + y - 1) = k \cdot y - (k - 1)$ so $\blacksquare_i x = y = \frac{x}{i} + \frac{i-1}{i}$
(ii) $x = y^i = min(y, min(y...min(y, y))...) = y$ i.e $\blacksquare_i x = x$
(iii)$x = y^i = y \cdot y \cdot ...y$ so we have $\blacksquare_i x = x^{\frac{1}{i}}$
▶

1.2 Some basic definitions and facts from algebra

We will begin with the definition of algebra, subalgebra, homomorphism image and Product then we will provide Birkhoff's theorem without proof and finally we will define for us important algebras.

Definition 1.2.1. A Structure of the form $\{M, f_1, f_2, ...f_n, ...\}$ where f_i-s are a operations on M is called *Algebra*.

Such an algebra is naturally a structure for a predicate language I having the equality predicate $=$, and function symbols $F_1, ..., F_n$ of corresponding arities. Many important classes of algebras are defined as classes of all models of theory T over I with the equality $=$. In particular, axioms, different from equality axioms may be just some atomic formulas, i.e. identities $t = s$ for some terms t and s. This leads to following:

Definition 1.2.2. Let I be language $(=, F_1, F_2, ...F_n...)$, let K be class of structures for I . K, is a variety if there is a set of T of identities such that K is a class of all structures M for I such that all identities from T are true in M

As an example of variety will serve a class of semigroups.

Definition 1.2.3. Let **M**= $\langle M, f_1, f_2, ...\rangle$ and **N** = $\langle N, g_1, g_2, ...\rangle$ be structures for the language I
(1) **M** is subalgebra of **N** if **M**$\subseteq$ **N** and for each i, f_i is restriction of g_i on **M**
(2) **M** is homomorphic image of **N** if there is a mapping h of **N** onto **M** commuting with operations, i.e. for each i and any $a_1, \ldots a_k \in M$, $h(f_i(a_1, ...a_k)) = g(h(a_1), ..., h(a_k))$ (where k is arity of f_i and g_i)
(3) Let C be non-empty set, and for each $\lambda \in C$ let $M_\lambda = \langle M_\lambda, f_{1\lambda}, f_{2\lambda}, ...\rangle$, be a structure for I, Direct product $\prod_{\lambda\in C} M_\lambda$ is the algebra $\mathbf{M} = \langle M, f_1, f_2, ...\rangle$ M is set of all function whose domain is C and for each $\lambda \in C$ $a(\lambda) \in M_\lambda$ (selectors). The operations are defined coordinatewise i.e. for each $(a_1, ...a_n) \in M$ and f_i , $f_i(a_1, ...a_k) = b$ iff for each $\lambda \in C$, $b(\lambda) = f_{i\lambda}(a_1(x), ...a_k(x))$. Note that in general non-emptines of **M** follows from set theoretical axiom of choice.

Theorem 1.2.4. *(**Birkhoff's theorem**) A class K of structures for I is a variety iff it is closed under subalgebras, homomorphic images and direct products, i.e. K contains with each element **M** its subalgebras and homomorphic images and will each system $\langle M_\lambda | \lambda \in C\rangle$ of elements of K contains its direct product $\prod_{\lambda\in C} M_\lambda$.*
◄ *see proof in [3]* ►

Now we will present definitions of different lattices and basic facts about them.

Definition 1.2.5. (**lattice**)The language of lattices has two binary function symbols $\cup$ and $\cap$, plus =. An algebra **L**=$\langle L, \cap, \cup\rangle$ is a lattice if the following identities are true in **L**:

$$
\begin{array}{lll}
x \cap x = x & x \cup x = x & \text{(idempotence)} \\
x \cap y = y \cap x & x \cup y = y \cup x & \text{(commutativity)} \\
x \cap (y \cap z) = (x \cap y) \cap z & x \cup (y \cup z) = (x \cup y) \cup z & \text{(associativity)} \\
x \cap (x \cup y) = x & x \cup (x \cap y) = x & \text{(absorption)}
\end{array}
$$

Remark 1.2.6. (1)Obviously class of lattices form a variety. (It is a class of all structures for **L**, where idempotence, commutativity, associativity and absorption (all this properties are expressed by identities) are true.)
(2) If we will take any linearly ordered set $\langle L, \leqslant\rangle$ and put $min(x, y)$ for $x \cap y$ and $max(x, y)$ for $x \cup y$, then $\langle L, \cap, \cup\rangle$ will be lattice.

Remark 1.2.7. If $\langle L, \cap, \cup\rangle$ is lattice, then $\langle L, \leqslant\rangle$ is ordered set where $\leqslant$ is defined as follows: $x \leqslant y$ means $x \cap y = x$, on the other hand let $\langle L, \leqslant\rangle$ be ordered set, put $inf(x, y)$ for $x \cap y$ and $sup(x, y)$ for $x \cup y$ then $\langle L, \cap, \cup \rangle$ is lattice. So we shall identify $\langle L, \cap, \cup \rangle$ with $\langle L, \cap, \cup, \leqslant \rangle$, where $\leqslant$ is defined as above.

We will summarize obvious properties of lattices which follows from the definition in following two lemma which we will give without proof. For proofs you can look in [4].

Lemma 1.2.8. *If* **L**$= \langle L, \cap, \cup, \leqslant \rangle$ *is a lattice then:*
(i) For any $a, b \in L$, $a \cap b = a$ *iff* $a \cup b = b$
(ii)

$$\begin{array}{rl} x \cap y \leqslant x & x \cap y \leqslant y \\ x \cup y \geqslant x & x \cup y \geqslant y \\ (\forall z)((z \leqslant x \wedge z \leqslant y) \to z \leqslant x \cap y) & (\forall z)((x \leqslant z \wedge y \leqslant z) \to x \cup y \leqslant z) \end{array}$$

(iii) $\cap$ *and* $\cup$ *are non-decreasing w.r.t.* $\leqslant$ *i.e.*

$$(x_1 \leqslant x_2 \wedge y_1 \leqslant y_2) \to (x_1 \cap x_2 \leqslant y_1 \cap y_2)$$
$$(x_1 \leqslant x_2 \wedge y_1 \leqslant y_2) \to (x_1 \cup x_2 \leqslant y_1 \cup y_2)$$

Definition 1.2.9. **L**$= \langle L, \cap, \cup, \leqslant \rangle$ lattice is linearly ordered lattice if $\leqslant$ is linear ordering.

Remark 1.2.10. Linearly ordered lattices is not a variety. Note that lattice **L** is linearly ordered iff $(x \cap y = x) \vee (x \cap y = y)$ is true in it. It is easy to prove that linearity can not be expressed by identities and i.e. that class of linearly ordered lattices is not a variety. By Birkhoff's theorem we can construct simple counterexample. Product of two linearly ordered algebras is not linearly ordered algebra.

Now we will define very important for us two class of algebras.

Definition 1.2.11. (**residuated lattice**) A residuated lattice is an algebra $\langle L, \cap, \cup, *, \Rightarrow, 0, 1 \rangle$ with four binary operations and two constants such that:
(i) $\langle L, \cap, \cup, 0, 1 \rangle$ is lattice with largest element 1 and least element 0 (with respect to the lattice ordering $\leqslant$)
(ii) $\langle L, *, 1 \rangle$ is a semigroup with unit element 1, i.e. $*$ is commutative, associative
and for all x , $x * 1 = x$
(iii)$*$ and $\Rightarrow$ form and adjoint pair i.e.

$$z \leqslant x \Rightarrow y \text{ iff } x * z \leqslant y$$

Definition 1.2.12. A residuated lattice is linearly ordered residuated lattice if lattice ordering $\leqslant$ is linear, i.e. for all $x, y \in L$ $x \cap y = x$ or $x \cap y = y$

Definition 1.2.13. (**BL-algebra**)The residuated lattice $\langle L, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra if following two identities hold for all $x, y \in L$

$$x \cap y = x * (x \Rightarrow y)$$
$$(x \Rightarrow y) \cup (y \Rightarrow x) = 1 \text{ } (\textit{Prelinearity})$$

Remark 1.2.14. Linearly ordered lattice is BL-algebra iff $x \cap y = x * (x \Rightarrow y)$

Theorem 1.2.15. *The class of all BL-algebras is a variety of algebras.*
◀ *By Remark 1.2.6 class of lattices is a variety, also conditions on 0 and 1 are expressible by identities ($x \cap 1 = x$, $x \cap 0 = 0$). Semigroup conditions on $*$ are also identities. To complete the proof only thing what should we do is to proof that adjointness condition can be expressed by identities. We will bring without proof Lemma (1.2.16) which claims that adjointness can be replaced by set of identities.*

Lemma 1.2.16. *If we will change in the definiton of BL-algebras adjointness condition by following identities*
(1) $x \cap (y \Rightarrow (x * y)) = x$,
(2) $((x \Rightarrow y) * x) \cup y = y$,
(3) $(x \Rightarrow (x \cup y)) = 1$,
(4) $((z \Rightarrow x) \Rightarrow (z \Rightarrow (x \cup y))) = 1$,
(5) $(x \cap y) * z = (x * z) \cap (y * z)$
then the class of BL-algebras will not change. In other words axioms of BL-algebra implies (1)... (5) identities and if we will replace adjointness condition by (1)... (5) identities then we can prove adjointness.
◀ *For the detailed proof see [1](Lemma 2.3.10)*▶
▶

Definition 1.2.17. (■-algebras) An algebra $\langle L, \cap, \cup, *, \Rightarrow, 0, 1, \blacksquare_1, \blacksquare_2, ...\rangle$ is ■-algebra if:
(1) $\langle L, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ is BL-algebra.
(2) For all $i \in \{1, 2, 3, ...\}$ and for all $x, y \in L$

$$(\blacksquare_i x)^i = x$$
$$y * (y^i \Rightarrow x) \leqslant \blacksquare_i x$$

Remark 1.2.18. Obviously ■-algebras form a variety. From the previous theorem we know that class of BL-algebras is a variety. $x \leqslant y$ means $min(x, y) = x$, so it means that all conditions in definition of ■-algebra, are expressed by identities, so class of ■-algebras is a variety.

Definition 1.2.19. A ■-algebra is linearly ordered ■-algebra if lattice ordering $\leqslant$ is linear, i.e. for all $x, y \in L$ $x \cap y = x$ or $x \cap y = y$

1.3 Ordered Abelian Groups

In this section we will define ordered Abelian groups and then we will give some examples and facts about them without proof.

Definition 1.3.1. (**Ordered Abelian Groups**) $\mathbf{G} = \langle G, \oplus, \underline{0}, -, \leqslant \rangle$ is ordered Abelian group iff following axioms are true in it:

$$\begin{gathered} x \oplus (y \oplus z) = (x \oplus y) \oplus z \\ x \oplus y = y \oplus x \\ x \oplus \underline{0} = x \\ x \oplus -x = \underline{0} \\ x \leqslant y \vee y \leqslant x \\ (x \leqslant y \wedge y \leqslant z) \rightarrow x \leqslant z \\ (x \leqslant y \wedge y \leqslant x) \rightarrow x = y \\ x \leqslant y \rightarrow (x \oplus z \leqslant y \oplus z) \end{gathered}$$

First four axioms are axioms of commutative or Abelian semigroup, next three are linear preorder axioms and the last one is monotonicity axiom which expresses connection between semigroup operation and preorder. 0 is zero element and $-x$ is called inverse of x

Example 1.3.2. **N**=$\langle N, +, \leqslant \rangle$, where N is set of natural numbers and $+$ and $\leqslant$ are addition and order on natural numbers respectively. Zero element is 0, no non-zero element has inverse.

Example 1.3.3. **N**=$\langle N, \bullet, \leqslant \rangle$, $\bullet$ and $\leqslant$ are multiplication and order on natural numbers respectively. Zero element is 1. no non-zero element has inverse.

Example 1.3.4. Most important example for us is **Re**=$\langle Re, +, 0, -, \leqslant \rangle$, where Re is set of real numbers and $+$ and $\leqslant$ are addition and order on it respectively. Zero element is 0.**Re** is called additive linearly ordered Abelian group of reals.

Example 1.3.5. **Z**=$\langle Z, +, \leqslant\rangle$, where Z is set of integer numbers and + and $\leqslant$ are addition and order on it respectively. Zero element is 0. **Z** is called additive linearly ordered Abelian group of integers and it is subalgebra of **Re**

Definition 1.3.6. Linearly ordered Abelian group is locally embeddable into **Re** if for each finite $x \subseteq G$ there is finite $y \subseteq Re$ and one-one mapping f of X onto Y which partial isomorphism, i.e. for each $x, y, z \in G$,

$$z = x \oplus y \text{ iff } f(z) = f(x) + f(y)$$
$$x \leqslant_G y \text{ iff } f(x) \leqslant f(y)$$

.

Now we will give Gurevich-Kokorin theorem without proof and direct consequence of it and then easy generalization of this consequence which we need for our aims.

Theorem 1.3.7. *(Gurevich-Kokorin) Let $\phi(x_1, ...x_n)$ be a quantifier free formula in the language of linearly ordered Abelian groups. If the formula $\forall x_1, ...x_n \phi(x_1, ...x_n)$ is true in linearly ordered Abelian group **Re** then it is true in all linearly ordered Abelian groups.*
◀ *see proof in [9]* ▶
As direct consequence of Gurevich-Kokorin theorem we have following:

Theorem 1.3.8. *Each lienarly ordered Abelian group is locally embeddable into **Re**.*
◀ *For detailed proof see [7](285 pp. Lemma 7.3.20)* ▶

Remark 1.3.9. We will need following easy generalization of previous theorem: we may assume that **G** is linearly ordered Abelian group with some additional operations $F_1, F_2, ...$ definable by open formulas from the group operation and ordering, i.e. there are open formulas ϕ_i such that

$$a = F_i(x_1, ...x_n) \leftrightarrow \phi_i(x_1, ...x_n, y)$$

is true **G** and **Re**. Partial isomorphism f can be found in such a way that it preserves all F_i.

Chapter 2

Completeness of Basic Many-valued Logic Extended By All Division Operators

In previous chapter we saw definitions of t-norms their residuas and divisibility witnesses and some basic properties of this three operations. We also introduced for us very important variety of algebras, $\blacksquare$-algebras. In the first section for all fixed continuous t-norm $*$ we will introduce corresponding propositional calculus $PC\Box(*)$, then we will formulate logical axioms, define provability and then we will show that basic logic extended by all division operators, BL$\Box$, is sound . In the second section we will provide proof of completeness of BL$\Box$ w.r.t $\blacksquare$-algebras.

2.1 The Basic Many-valued Logic Extended by All Division Operators

For all fixed continuous t-norm $*$ we can define a propositional calculus.

Definition 2.1.1. The propositional calculi PC$\Box(*)$ given by $*$ has:
1. Propositional Variables: $p_1, p_2, p_3...$
2. Constant: $\overline{0}$
3. Connectives: two binary-connectives & and $\rightarrow$ and infinite number of unary-connectives $\Box_1, \Box_2, \Box_3...$
Formula is defined in the obvious way:
1. Propositional variable is a Formula, $\overline{0}$ is a Formula.

2. If ϕ is a Formula then $\Box_i\phi$ is also Formula $\forall i, i \in \{1,2,3...\}$
3. If ϕ and ψ are Formulas than $\phi\&\psi$ and $\phi \to \psi$ are Formulas .

Intuitive meaning of connectives & and$\to$ connectives are well known and $\Box_i\phi$ intuitively is maximal solution of $\underbrace{\psi\&\ldots\&\psi}_{i\ times} \leftrightarrow \phi$ and its truth function is $\blacksquare_i$

Definition 2.1.2. (**Evaluation of Formulas**) Evaluation of propositional variables is a mapping e assigning to each propositional variable p its truth value $e(p) \in [0,1]$. If e is an evaluation of propositional variables than evaluation of all Formulas is defined as follows:
$e(\overline{0}) = 0$
$e(\Box_i\phi) = \blacksquare_i e(\phi)\ \forall i, i \in \{1,2,3...\}$
$e(\phi\&\psi) = e(\phi) * e(\psi)$
$e(\phi \to \psi) = e(\phi) \Rightarrow e(\psi)$
where $\Rightarrow$ and $\blacksquare_i$ respectively are residuum and divisibility witness of t-norm $*$.

Definition 2.1.3. (**short notations**)
$\overline{1}$ is $\overline{0} \to \overline{0}$
$\phi \wedge \psi$ is $\phi\&(\phi \to \psi)$
$\phi \vee \psi$ is $((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$
$\neg\phi$ is $\phi \to \overline{0}$
$\phi \leftrightarrow \psi$ is $(\phi \to \psi)\&(\psi \to \phi)$
ϕ^n is $\underbrace{\phi\&\ldots\&\phi}_{n\ times}$

Remark 2.1.4. Because of 1.1.7 it is easy to check that the truth functions of $\wedge$ and $\vee$ are respectively *min* and *max*.

Definition 2.1.5. Basic logic extended by all Division operators we will denote by $BL\Box$. Following are axioms of $BL\Box$
$(A1)$ $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ transitivity of implication
$(A2)$ $(\phi\&\psi) \to \phi$ conjunction implies first conjunct
$(A3)$ $(\phi\&\psi) \to (\psi\&\phi)$ commutativity of conjunction
$(A4)$ $(\phi \wedge \psi) \to (\psi \wedge \phi)$ commutativity of $\wedge$
$(A5)$ $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi\&\psi) \to \chi)$
$(A6)$ $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$ variant of prove by cases
$(A7)$ $\overline{0} \to \phi$. $\overline{0}$ implies everything.

For all $i \in \{1, 2, 3...\}$ we have following Axioms:
$(A8_i)(\Box_i\phi)^i \leftrightarrow \phi$.
$(A9_i)(\psi^i \to \phi) \to (\psi \to \Box_i\phi)$

Remark 2.1.6. $(A5)$ express residuation. Meaning of $(A8)$ is clear and $(A9)$ express that $\Box_i\phi$ is maximal from all ψ, such that $\psi^i \to \phi$.

Remark 2.1.7. As We can see we have infinite number of connectives and respectively infinite number of axioms in $BL\Box$.

The Deduction rule for $BL\Box$ is Modus Ponens.

Definition 2.1.8. $P_1, P_2, \ldots, P_n$ is a proof of $BL\Box$ formula ϕ if $P_n = \phi$ and for all $i \in \{1, 2, \ldots n\}$, P_i is either axiom of $BL\Box$ or there exists $m, k < i$ such that P_i follows from P_m and P_k by Modus Ponens .

Definition 2.1.9. ϕ is provable, if there exist proof of ϕ and will denote this by $\vdash \phi$, or $BL\Box \vdash \phi$.

Now we will define a 1-tautology in PC$\Box(*)$ and will prove that all provable formulas are 1-tautologies. In next section we will show other direction of this implication.

Definition 2.1.10. A Formula ϕ is 1-tautology of PC$\Box(*)$ if $e(\phi) = 1$ for every evaluation e.

Theorem 2.1.11. ***(soundness)*** *If $BL\Box \vdash \phi$ then ϕ is 1-tautology.*
◀ *It is easy to see that if ϕ and $\phi \to \psi$ are 1-tautologies then ψ is also 1-tautology. So now it is enough to show that all axioms of $BL\Box$ are 1-tautologies.*
$(A1)$ $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$, *let us assume that $e(\phi) = x, e(\psi) = y, e(\chi) = z$, it is enough to show that $1 \leqslant (x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))$ by residuation it is true iff $(x \Rightarrow y) \leqslant ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))$ is true, and again by residuation, this is true iff $(x \Rightarrow y) * (y \Rightarrow z) \leqslant (x \Rightarrow z)$ is true, and again by residuation it is true iff $(x \Rightarrow y)*(y \Rightarrow z)*x \leqslant z$. But since we know $x*(x \Rightarrow y) = min(x, y) \leqslant y$ and similarly $y*(y \Rightarrow z) = min(y, z) \leqslant z$,*

$(x \Rightarrow y) * (y \Rightarrow z) * x \leqslant z$ is true and it means that (A1) is 1-tautology.
$A2, A3, A4, A7$ and $A8$ follows immediately from definitions.
($A5$) By residuation we have: $t \leqslant x \Rightarrow (y \Rightarrow z)$ iff $t * x \leqslant y \Rightarrow z$ iff $t * x * y \leqslant z$ iff $t \leqslant x * y \Rightarrow z$ and from this equivalence it is easy to see that $A5$ is 1-Tautology.
($A6$) Observe that $x \Rightarrow y = 1$ or $y \Rightarrow x = 1$ and that $1 \Rightarrow y = y$, if we have $x \Rightarrow y = 1$ than $A6$ is equivalent to following $z \Rightarrow (t \Rightarrow z)$ which obviously is a 1-Tautology. If $y \Rightarrow x = 1$ then $A6$ is equivalent to following $t \Rightarrow 1$, so $A6$ is 1-Tautology.
($A9$) Follows from $y * (y^i \Rightarrow x) \leqslant \blacksquare_i x$ and residuation ▶

2.2 Completeness

So our goal in this section is to prove the completeness theorem in the following form:
***completeness theorem**"For each formula ϕ the following three things are equivalent:*
(i) ϕ is provable in BL□
*(ii) For each linearly ordered ■-algebra **L**, ϕ is an **L**-tautology.*
*(iii)For each ■-algebra **L**, ϕ is an **L**-tautology."*
Let us comment why did we choose this form of completeness, when only equivalence of (i) and (iii) would be enough and has much easier proof then equivalence of (i), (ii) and (iii). The thing is that (ii)⟶(iii) implication is needed to prove completeness theorem for Propositional Lukasiewicz , Gödel and Product Logics extended by all division operators.
Obviously for each t-norm $*$, unit interval [0,1] endowed with residua and divisibility witnesses of $*$ is ■-algebra, even more it is linearly ordered ■-algebra. from this fact and completeness theorem above we will get completeness for 1-tautologies.

Theorem 2.2.1. *For each t-norm $*$, $\langle [0,1], \cap, \cup, *, \Rightarrow, 0, 1, \blacksquare_1, \blacksquare_2, ... \rangle$, where $\Rightarrow$ is residua of $*$ and $\blacksquare_1, \blacksquare_2, ...$ are divisibility witnesses of $*$, is linearly ordered ■-algebra.*
◀ *obvious* ▶

Now we are on proper place to begin proof of completeness. This proof has following steps:
*(i)We shall introduce for each ■-algebra **L**, notion of **L**-tautology which is generalization of 1-tautology, and shall prove that BL□ is sound over each*

linear $\blacksquare$*-algebra, i.e. BL*$\Box$*-provable formula is* ***L****-tautology over each linear* $\blacksquare$*-algebra.*
(ii) We will prove that set of classes of provably equivalent formulas, endowed with operations given by &, $\rightarrow$*,* $\Box_i$ *, is a* $\blacksquare$*-algebra.*
(iii) Then we will prove that if formula is tautology over each linearly ordered $\blacksquare$*-algebra then it is also tautology over each* $\blacksquare$*-algebra.*

Definition 2.2.2. (**L-evaluation and L-tautology**)Let **L**=$\langle L, \cap, \cup, *, \Rightarrow, 0, 1, \blacksquare_1, \blacksquare_2, ...\rangle$ be a $\blacksquare$-algebra.
(i) **L**-Evaluation of propositional variables is a mapping e assigning to each propositional variable p its truth value $e(p) \in L$. If e is **L**-evaluation of propositional variables than evaluation of all Formulas is defined as follows:
$e(\overline{0}) = 0$
$e(\Box_i\phi) = \blacksquare_i e(\phi)$ $\forall i, i \in \{1, 2, 3...\}$
$e(\phi \& \psi) = e(\phi) * e(\psi)$
$e(\phi \rightarrow \psi) = e(\phi) \Rightarrow e(\psi)$
where $\Rightarrow$ and $\blacksquare_i$ respectively are residuum and divisibility witness of t-norm $*$.
(ii) BL$\Box$ formula ϕ is a **L**-tautology if under each **L**-evaluation e, e(ϕ)=1.

Theorem 2.2.3. *If* $BL\Box \vdash \phi$ *then* ϕ *is* ***L****-tautology for all linearly ordered* $\blacksquare$*-algebra.*
◀ *It is clear that the deduction rule (Modus Ponens) is sound. The proof of axioms are exactly same as it was in the proof of* ***Theorem 2.1.11****. Only thing we have to show is that definition of* $x \cup y$ *is* ***L****-tautology.*
$((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x) =$
$=(((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)) * ((x \Rightarrow y) \cup (y \Rightarrow x))=$ *(because* $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$ *)*
$=((((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)) * (x \Rightarrow y)) \cup ((((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)) * (y \Rightarrow x))$
$\leqslant ((x \Rightarrow y) \Rightarrow y) * (x \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x) * (y \Rightarrow x) \leqslant y \cup x = x \cup y$
On the other hand $(x \Rightarrow y)*(x \cup y) = (x*(x \Rightarrow y)) \cup (x*(x \Rightarrow y) \leqslant y \cup y = y$ *and by residuation* $x \cup y \leqslant (x \Rightarrow y) \Rightarrow y$*, analogously* $x \cup y \leqslant (y \Rightarrow x) \Rightarrow x$*, and hence* $x \cup y \leqslant ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$ ▶

Remark 2.2.4. In the proof of previous theorem we used linearity only when we proved (A6), but we can prove it just with prelinearity, so it means that: If $BL\Box \vdash \phi$ then ϕ is **L**-tautology for all $\blacksquare$-algebra **L** is true. We are going to prove (ii)$\longrightarrow$(iii) implication, so proof of (A6) without linearity is needless.

Definition 2.2.5. Let T be a fixed theory over BL□. For each formula ϕ let define $[\phi]_T = \{\psi | T \vdash \phi \leftrightarrow \psi\}$. L_T is a set of all the classes $[\phi]_T$. We define :

$$0 = [\overline{0}]_T$$
$$1 = [\overline{1}]_T$$
$$\blacksquare_i[\phi]_T = [\Box_i\phi]_T$$
$$[\phi]_T * [\psi]_T = [\phi \& \psi]_T$$
$$[\phi]_T \Rightarrow [\psi]_T = [\phi \to \psi]_T$$
$$[\phi]_T \cap [\psi]_T = [\phi \wedge \psi]_T$$
$$[\phi]_T \cup [\psi]_T = [\phi \vee \psi]_T$$

This algebra is denoted by $\mathbf{L}_T$. (It is not difficult to check that this definitions are correct).

Lemma 2.2.6. *$\boldsymbol{L}_T$ is a BL-algebra.*
◀ *See proof in [1](Lemma 2.3.12)* ▶

Lemma 2.2.7. *(i)L_T is ■-algebra.*
(ii) $L_{BL\Box}$ is ■-algebra.
◀
(i) By previous lemma we need only check that the following two axioms of ■-algebra 1.$(\blacksquare_i[\phi]_T)^i = [\phi]_T$ and 2. $[\psi]_T * ([\psi]_T^i \Rightarrow [\phi]_T) \leqslant \blacksquare_i[\phi]_T$ are true in $\boldsymbol{L}_T$.
1. $(\blacksquare_i[\phi]_T)^i = \underbrace{\blacksquare_i[\phi]_T * \ldots * \blacksquare_i[\phi]_T}_{i\ times} = \underbrace{(\Box_i[\phi])_T * \ldots * (\Box_i[\phi])_T}_{i\ times} = (\underbrace{\Box_i[\phi] \& \ldots \& \Box_i[\phi]}_{i\ times})_T$
$= ((\Box_i[\phi])^i)_T = [\phi]_T$

2. Since $\boldsymbol{L}_T$ is BL-algebra, by residuation $[\psi]_T * ([\psi]_T^i \Rightarrow [\phi]_T) \leqslant \blacksquare_i[\phi]_T$ iff $[\psi]_T^i \Rightarrow [\phi]_T \leqslant [\psi]_T \Rightarrow \blacksquare_i[\phi]_T$.
By definition of $\boldsymbol{L}_T$, $[\psi]_T^i \Rightarrow [\phi]_T = (\psi^i \to \phi)_T$ and $[\psi]_T \Rightarrow \blacksquare_i[\phi]_T = (\psi \to \Box_i\phi)_T$, so we have
$[\psi]_T^i \Rightarrow [\phi]_T \leqslant [\psi]_T \Rightarrow \blacksquare_i[\phi]_T$ iff $(\psi^i \to \phi)_T \leqslant (\psi \to \Box_i\phi)_T$.
Now observe that: $[\phi]_T \leqslant [\psi]_T$ iff $T \vdash \phi \to \psi$ so
$[\psi]_T * ([\psi]_T^i \Rightarrow [\phi]_T) \leqslant \blacksquare_i[\phi]_T$ iff $T \vdash (\psi^i \to \phi) \to (\psi \to \Box_i\phi)$
Since $(\psi^i \to \phi) \to (\psi \to \Box_i\phi)$ is the axiom (A9) of BL□, it is provable, so proof of (i) is completed.
(ii) It is subcase of (i), $T = \emptyset$ ▶

Definition 2.2.8. Let $\mathbf{L} = \langle L, \cap, \cup, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. A filter on $\mathbf{L}$ is non-empty set $F \subseteq L$ such that for all $x, y \in L$,

$$a \in F \text{ and } b \in F \text{ implies } a * b \in F$$
$$a \in F \text{ and } a \leqslant b \text{ implies } b \in F$$

F is a prime filter iff for each $x, y \in L$,

$$x \Rightarrow y \in F \text{ or } y \Rightarrow x \in F$$

Lemma 2.2.9. *Let* $\boldsymbol{L}$ *be a BL-algebra and Let* F *be a filter. Put*

$$x \sim_F y \text{ iff } x \Rightarrow y \in F \text{ and } y \Rightarrow x \in F$$

Then :
(i) $\sim_F$ *is congruence and the corresponding quotient algebra* $\boldsymbol{L}/\sim_F$ *is BL-algebra.*
(ii)$\boldsymbol{L}/\sim_F$ *is linearly ordered iff* F *is prime filter.*
◀ *See proof in [1](Lemma 2.3.14)* ▶

Obviously as a corollary of previous lemma we have similar result about ■*-algebras:*

Corollary 2.2.10. *Let* $\boldsymbol{L}$ *be a* ■*-algebra and Let* F *be a filter. Put*

$$x \sim_F y \text{ iff } x \Rightarrow y \in F \text{ and } y \Rightarrow x \in F$$

Then :
(i) $\sim_F$ *is congruence and the corresponding quotient algebra* $\boldsymbol{L}/\sim_F$ *is* ■*-algebra.*
(ii)$\boldsymbol{L}/\sim_F$ *is linearly ordered iff* F *is prime filter.*
◀ *(i) In the proof of previous lemma is proved that assigning to each* x *its class* $[x]_F$ *is homomorphism. Since* ■*-algebras form a variety, by Birkhoff's theorem* $\boldsymbol{L}/\sim_F$ *is also* ■*-algebra.*
(ii) Exactly same as (ii) of previous lemma ▶

Lemma 2.2.11. *Let* $\boldsymbol{L}$ *be a BL-algebra and let* $a \in L$ *,* $a \neq 1$*, then there is a prime filter* F *on* $\boldsymbol{L}$ *not containing* a
. ◀ *See proof in [1](Lemma 2.3.15)* ▶

Corollary 2.2.12. *Let* $\boldsymbol{L}$ *be a* ■*-algebra and let* $a \in L$ *,* $a \neq 1$*, then there is a prime filter* F *on* $\boldsymbol{L}$ *not containing* a *.*

Lemma 2.2.13. *(**Subdirectproduct representation**)Each $\blacksquare$-algebra $\boldsymbol{L}$ is (isomorphic to) a subalgebra of the direct product of a system of linearly ordered $\blacksquare$-algebras .*
◀ Let $\mathcal{U}$ be a set of all prime filters F on $\boldsymbol{L}$ and for all $F \in \mathcal{U}$ define $\boldsymbol{L}_F = \boldsymbol{L}/F$. Let

$$L^* = \prod_{F \in \mathcal{U}} \boldsymbol{L}_F$$

From (ii) of corollary 2.2.10, L^* is a product of linearly ordered $\blacksquare$-algebras $\{\boldsymbol{L}_F \mid F \in \mathcal{U}\}$. For all $x \in L$ define $i(x) = \{[x]_F \mid F \in \mathcal{U}\}$. From (i) of corollary 2.2.10 it is clear that this embedding preserves operations. It remains to show that i is injective (one to one embedding). If $x, y \in F$ and $x \neq y$ then $x \nleq y$ or $x \ngeq y$. Assume $x \nleq y$, so $x \Rightarrow y \neq 1$ in $\boldsymbol{L}$. By corollary 2.2.12 there exist prime filter F not containing $x \Rightarrow y$, so $x \nsim_F y$, hence $[x]_F \neq [y]_F$ and it means that $i(x) \neq i(y)$. The case $x \ngeq y$ is completely same. So we proved that if $x \neq y$ then $i(x) \neq i(y)$ and this means that i is injective. So i is homomorphism (from $\boldsymbol{L}$ to L^*) and $i(\boldsymbol{L})$ is subalgebra of L^* which is product of a system of linearly ordered $\blacksquare$-algebras and $\boldsymbol{L}$ is isomorphic to $i(\boldsymbol{L})$. ▶

Definition 2.2.14. Associate with each formula ϕ of BL$\Box$ a term $\phi^\bullet$ of language of $\blacksquare$-algebras by replacing the connectives $\rightarrow, \&, \wedge, \vee, \overline{0}, \overline{1}, \Box_1, \Box_2, \Box_3 \ldots$ by function symbols and constants $\Rightarrow, *, \cap, \cup, 0, 1, \blacksquare_1, \blacksquare_2, \blacksquare_3 \ldots$ respectively and replacing each propositional variable p_i by corresponding object variable x_i .

Lemma 2.2.15. *(i) For all $\blacksquare$-algebras $\boldsymbol{L}$ and for all formulas ϕ of BL$\Box$, ϕ is a $\boldsymbol{L}$-tautology iff $\phi^\bullet = 1$ is true in $\boldsymbol{L}$*
(ii)Each formula which is an $\boldsymbol{L}$-tautology for all linearly ordered $\blacksquare$-algebras is an $\boldsymbol{L}$-tautology for all $\blacksquare$-algebras.
◀ (i) If we will compare definitions of $\boldsymbol{L}$-evaluation and $\boldsymbol{L}$-tautology with definition of $\phi^\bullet$, then it will be clear that ϕ is a $\boldsymbol{L}$-tautology iff $\phi^\bullet = 1$ is true in $\boldsymbol{L}$
(ii) Immediately follows from subdirectproduct representation and (i) . ▶

Theorem 2.2.16. *(**Completeness**)For each formula ϕ the following three things are equivalent:*
(i) ϕ is provable in BL$\Box$
(ii) For each linearly ordered $\blacksquare$-algebra $\boldsymbol{L}$, ϕ is an $\boldsymbol{L}$-tautology.
(iii)For each $\blacksquare$-algebra $\boldsymbol{L}$, ϕ is an $\boldsymbol{L}$-tautology.
◀
*(i)$\longrightarrow$ (ii) is exactly **Theorem 2.2.3***

*(ii)$\longrightarrow$ (iii) is exactly (ii) of **Lemma 2.2.15***
*(iii)$\longrightarrow$ (i) . Assume (iii). By (ii) of **Lemma 2.2.7** $L_{BL\Box}$ is also $\blacksquare$-algebra, so ϕ is $L_{BL\Box}$-tautology. Let e be an $L_{BL\Box}$-evaluation then, $e(\phi) = [\phi]_{BL\Box} = [1]_{BL\Box}$ and it means that $BL\Box \vdash \phi \leftrightarrow 1$ and finally we get $BL\Box \vdash \phi$. This completes the proof.*
►

For our future aims we will need one generalization of previous theorem for this we need following:

Definition 2.2.17. (i) An axiom schema given by a formula $\Phi(p_1, ..., p_n)$ is the set of all formulas $\Phi(\phi_1, ..., \phi_n)$ resulting by the substitution of ϕ_i for p_i $(i = 1, 2, ..., n)$ in $\Phi(p_1, ..., p_n)$.
(ii) A logical calculus $\mathcal{C}$ is a schematic extension of BL□ if it results from BL□ by adding some axiom schemata to its axioms. (The deduction rule remains to be Modus Ponens)
(iii) Let $\mathcal{C}$ be a schematic extension of BL□ and let **L** be ■-algebra. **L** is $\mathcal{C}$-algebra if all axioms of $\mathcal{C}$ are **L**-tautologies.

Theorem 2.2.18. ***(Completeness)****Let $\mathcal{C}$ be a schematic extension of BL□. For each formula ϕ the following three things are equivalent:*
(i) $\mathcal{C}$ proves ϕ
*(ii) For each linearly ordered $\mathcal{C}$-algebra **L**, ϕ is an **L**-tautology.*
*(iii)For each $\mathcal{C}$-algebra **L**, ϕ is an **L**-tautology.*
◄ *proof is analogous to proof of previous theorem* ►

Chapter 3

Completeness of Łukasiewicz Logic Extended by All Division Operators

Our aim in this chapter is to show that Łukasiewicz logic extended by all division operators is complete with respect to 1-tautologies over [0,1].

3.1 Łukasiewicz Logic extended by All Division Operators

As we know propositional logic given by Łukasiewicz t-norm is axiomatized by BL+(¬¬), where (¬¬) stands for double negation axiom

$$\neg\neg\phi \rightarrow \phi \ (\neg\neg)$$

Respectively Łukasiewicz logic extended by all division operators, which we will denote by Ł+□ is axiomatized by BL□+(¬¬) So axioms of Łukasiewicz logic extended by all division operators are, axioms of BL + □ + ¬¬ where by □ is denoted following axioms of division operator:

$$\text{For all } i \in \{1, 2, 3...\}$$
$$(\Box_i \phi)^i \leftrightarrow \phi \ .$$
$$(\psi^i \rightarrow \phi) \rightarrow (\psi \rightarrow \Box_i \phi)$$

Analogous to [1], we also will denote BL+(¬¬) by Ł, so BL□+(¬¬) and Ł+□ are same set of axioms and this set of axioms with Modus Ponens as a deduction rule is an axiomatization of Łukasiewicz logic extended by all *division operators.*

Let us give now more elegant characterization of Ł and respectively of Ł+□. For this we will define original set of Łukasiewicz axioms and then will give lemma without prove, which states that BL+(¬¬) = Ł is equivalent to Ł', where Ł' is original set of Łukasiewicz axioms.

Definition 3.1.1. Following are Łukasiewicz axioms :
(Ł1) $\phi \to (\psi \to \phi)$
(Ł2) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$
(Ł3) $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$
(Ł4) $((\phi \to \psi) \to \psi) \to ((\psi \to \phi) \to \phi)$
Set of Łukasiewicz axioms we will denote by Ł'.

Lemma 3.1.2. *(i)Ł proves all axioms of Ł'*
(ii) (Ł') proves all axioms of Ł
◀ *see proof in [1](subsection 3.1))*▶

So from this we can conclude that Ł+□ and Ł'+□ are equivalent set of axioms and both of them are axiomatization of Łukasiewicz logic extended by all division operators.

3.2 Completeness

By definition 2.2.17 Ł+□ is a schematic extension of BL□ and for this schematic extension holds completeness theorem 2.2.18 with respect to Ł+□-algebras i.e. BL□ algebras where $x = ((x \Rightarrow 0) \Rightarrow 0)$ *is valid.*

Definition 3.2.1. (**MV-algebras**) MV-algebra is BL-algebra where $x = ((x \Rightarrow 0) \Rightarrow 0)$ is valid.

Remark 3.2.2. It is easy to see that Algebra $\langle L, \cap, \cup, *, \Rightarrow, 0, 1, \blacksquare_1, \blacksquare_2, ...\rangle$ is Ł+□-algebra if $\langle L, \cap, \cup, *, \Rightarrow, 0, 1, \rangle$ is MV-algebra and $(\blacksquare_i x)^i = x$ and $y * (y^i \Rightarrow x) \leqslant \blacksquare_i x$ for all i are valid in it.

Remark 3.2.3. If we will show that each 1-tautology is also tautology over all linearly ordered Ł+□-algebras from completeness theorem above we will get desired completeness with respect to 1-tautologies.

Let us give another characterization of MV-algebras which is based on original Łukasiewicz axioms.

Definition 3.2.4. (**Wajsberg algebras**) Wajsberg algebra is an algebra $A = \langle A, \Rightarrow, 0\rangle$ in which following identities are valid.
$(W1)$ $(1 \Rightarrow y) = y$
$(W2)$ $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z))$
$(W3)$ $((-x \Rightarrow -y) \Rightarrow (y \Rightarrow x)) = 1$
$(W4)$ $((x \Rightarrow y) \Rightarrow y) = ((y \Rightarrow x) \Rightarrow x)$
where $-x$ is $x \Rightarrow 0$ and 1 is $0 \Rightarrow 0$

Now we will provide a theorem without proof which is about connection of Wajsberg algebras and MV-algebras.

Theorem 3.2.5. *(i) If $\langle A, \cap, \cup, *, \Rightarrow, 0, 1, \rangle$ is MV-algebra then $\langle A, \Rightarrow, 0\rangle$ is Wajsberg algebra.*
(ii) If $\langle A, \Rightarrow, 0\rangle$ is Wajsberg algebra then $\langle A, \cap, \cup, *, \Rightarrow, 0, 1, \rangle$ is MV-algebra where $*, \cup, \cap$ and 1 are defined in the following way:

$$x * y = -(x \Rightarrow -y) \text{ where } -x \text{ is } x \Rightarrow 0$$
$$x \cap y = x * (x \Rightarrow y)$$
$$x \cup y = (x \Rightarrow y) \Rightarrow y$$

◀ *see proof in [1](Theorem 3.2.7))*▶

Now we will give a famous characterization of linearly ordered MV-algebras by linearly ordered Abelian groups and after this we will extend this to the characterization of linearly ordered Ł+□-algebras.

Definition 3.2.6. Let $G = \langle G, \oplus, \ominus, 0_G, \leqslant_G\rangle$ be a linearly ordered Abelian group and let $e \in G, 0 < e$ be a positive element. $MV(G, e)$ is the algebra $\mathbf{A}= \langle A, \Rightarrow, 0_G\rangle$ whose domain A is the interval $[0, e]_G = \{x \in G | 0 \leqslant_G x \leqslant_G e\}$, $\Rightarrow$ is defined as follows: if $x \leqslant_G y$ then $x \Rightarrow y = e$ else $x \Rightarrow y = e \ominus x \oplus y$.

Lemma 3.2.7. *MV(G,e) is linearly ordered MV-algebra.*
◀ *see proof in [1](Lemma 3.2.10))*▶

Example 3.2.8. MV(**Re**,1) is standard MV-algebra [0,1].

Definition 3.2.9. Let $G = \langle G, \oplus, \ominus, 0_G, \leqslant_G \rangle$ be a linearly ordered Abelian group and let $e \in G, 0 < e$ be a positive element. $MV\blacksquare(G, e)$ is the algebra **A**$= \langle A, \Rightarrow, 0_G, \blacksquare_1, \blacksquare_2 ... \rangle$ whose domain A is the interval $[0, e]_G = \{x \in G | 0 \leqslant_G x \leqslant_G e\}$, $\Rightarrow$ is defined as follows: if $x \leqslant_G y$ then $x \Rightarrow y = e$ else $x \Rightarrow y = e \ominus x \oplus y$ and for all i following holds:
$(\blacksquare_i x)^i = x$
$y * (y^i \Rightarrow x) \leqslant \blacksquare_i x.$
where $*$ and $\leqslant$ are defined from $\Rightarrow$ in the obvious way.

Remark 3.2.10. Obviously if **A**$= \langle A, \Rightarrow, 0_G, \blacksquare_1, \blacksquare_2 ... \rangle$ is $MV\blacksquare(G, e)$ then **A'**$= \langle A, \Rightarrow, 0_G \rangle$ is $MV(G, e)$.

Remark 3.2.11. From Remark 3.2.2 and previous Lemma it is clear that $MV\blacksquare(G, e)$ is linearly ordered Ł+□-algebra.

Example 3.2.12. MV■(**Re**,1) is standard Ł+□-algebra [0,1].

Lemma 3.2.13. *For each linearly ordered MV-algebra* ***A*** *there is a linearly ordered Abelian group* ***G*** *and its positive element e such that* ***A***$=MV(G, e)$.
◀ *see proof in [1](Theorem 3.2.11))*▶

Corollary 3.2.14. *For each linearly ordered Ł+□-algebra* ***A*** *there is linearly ordered Abelian group* ***G*** *and its positive element e such that* ***A***$=MV\blacksquare(G, e)$.
◀ *Assume* ***A***$=\langle A, \cap, \cup, *, \Rightarrow, 0, 1, \blacksquare_1, \blacksquare_2, ... \rangle$*.From the linearity of* ***A*** *and Remark 3.2.2 we have that* ***A'***$=\langle A, \cap, \cup, *, \Rightarrow, 0, 1, \rangle$ *is a linearly ordered MV-algebra. By previous Lemma, there exists linearly ordered Abelian group* ***G'*** *and its positive element* e' *such that* ***A'***$=MV(G', e')$*. Let us extend the signature of* ***A'*** *with unary function symbols* $\blacksquare'_1, \blacksquare'_2, ...$ *and define for each i* $\blacksquare'_i$ *as a maximal solution of equation* $y^i = x$*. Extended* ***A'*** *we will denote by* ***A"****. Since* $(\blacksquare_i x)^i = x$ *and* $y * (y^i \Rightarrow x) \leqslant \blacksquare_i x$ *expresses that* $\blacksquare_i x$ *is maximal solution of equation* $y^i = x$*, we have* ***A"***$=MV\blacksquare(G', e')$ *. It is easy to see that for all i,* $\blacksquare_i$ *is completely defined by* $*$*, but in* ***A****,* ***A'*** *and* ***A"****,* $*$ *are the same, so for all i,* $\blacksquare_i$ *from* ***A*** *and* $\blacksquare'_i$ *from* ***A"*** *are same and it means that* ***A***$=$***A"***$=MV\blacksquare(G', e')$*.* ▶

Lemma 3.2.15. *If an identity* $\sigma = \tau$ *in the language of Ł+□-algebras is valid in standard Ł+□-algebra [0,1] with truth functions, then it is valid in each linearly ordered Ł+□-algebra.*
◀ *By Gurevich-Kokorin theorem* $\forall$*-sentence of ordered Abelian groups is true in the additive ordered Abelian group* ***Re*** *iff it is true in all linearly ordered Abelian groups. By remark 1.3.9 same is true if we introduce new*

operations by open definitions. In particular expand the theory of ordered Abelian groups by following operations:
$x \Rightarrow_e y = e$ *if* $x \leqslant y$*, otherwise* $x \Rightarrow_e y = e - x + y$
$-_e x = x \Rightarrow_e 0$
$x *_e y = -(x \Rightarrow_e -y)$
$x \cap_e y = x *_e (x \Rightarrow_e y)$
$x \cup_e y = (x \Rightarrow_e y) \Rightarrow_e y$
$\blacksquare_i^e x = y$ *if* $\underbrace{y *_e ... *_e y}_{i\ times} = x$ *and* $(\underbrace{z *_e ... *_e z}_{i\ times} = x) \longrightarrow z \leqslant y$
Now for each term σ *of Ł+□-algebras (wlog assume that* σ *is constructed only using variables and* $0, \Rightarrow,$ *and* $\blacksquare_i$ $i = 1, 2, ...$*) construct a term* σ_e^* *of ordered Abelian groups putting* $x_{i_e}^* = x_i$, $0^* = 0$, $(\sigma \rightarrow \tau)_e^* = \sigma_e^* \Rightarrow_e \tau_e^*$, $\blacksquare_i^e \sigma = \blacksquare_i^e \sigma_e^*$ *. Let* ***A*** *be linearly ordered Ł+□-algebra and* σ *and* τ *terms of* ***A*** *such that* $\sigma = \tau$ *is not valid in* ***A****, i.e. for some* $\bar{a} = a_1, ...a_n \in$***A****,* ***A***$\models \sigma(\bar{a}) \neq \tau(\bar{a})$ *. Since* ***A*** *is a linearly ordered Ł+□-algebra by corollary 3.2.14, there exists linearly ordered Abelian Group* ***G*** *and* $e \in G$ *s.t.* ***A***$=MV\blacksquare(G, e)$*. Thus* ***G***$\models \sigma_e^*(\bar{a}) \neq \tau_e^*(\bar{a})$ *and* ***G***$\models 0 \leqslant \bar{a} \leqslant e$*. By Gurevich-Kokorin theorem, there are real numbers* $e, a_1, ...a_n$ *s.t.* $0 < a_1, ...a_n < e$ *and* ***Re***$\models \sigma_e^*(\bar{a}) \neq \tau_e^*(\bar{a})$*, (*$\bar{a} = a_1, ...a_n$*) by dividing by e, we get* $b_1, ...b_n$ *such that* ***Re***$\models \sigma_1^*(\bar{b}) \neq \tau_1^*(\bar{b})$*, and it means that standard Ł+□-algebra [0,1] satisfies* $\sigma(\bar{b} \neq \tau(\bar{b})$ *and this completes our proof.* ▶

As a corollary of previous Lemma we have following:

Corollary 3.2.16. *If a formula* ϕ *is 1-tautology over the standard Ł+□-algebra [0,1], then it is* ***A****-tautology over each linearly ordered Ł+□-algebra* ***A****.*

Now if we will combine previous corollary with completeness theorem 2.2.18 we will get desired completeness of Ł+□ with respect to 1-tautologies.

Theorem 3.2.17. ***(Completeness)*** *A formula* ϕ *is 1-tautology of Łukasiewicz Logic extended by all division operators Ł+□, iff* ϕ *is provable in Ł+□.*
◀ *Follows immediately from remark 3.2.3 and previous corollary* ▶

Chapter 4

Propositional Quantifier Elimination For Quantified Propositional Fuzzy Logics

In this section we will provide definitions of propositional quantifiers, quantifier elimination, interpolation and uniform interpolation. Then we will give some examples of Quantified Propositional Fuzzy Logics which admits quantifier elimination.

4.1 Propositional Quantifiers

We will work in the language of propositional Logic which contains:
1. Infinite set of variable $V = \{x, x_1, x_2 ..., y, y_1, y_2, ...\}$
2. Constants 0 and 1
3. Connectives $\&, \rightarrow$
Formulas and substitutions are defined in the obvious way. As a truth value set we will take some W *such that* $\{0, 1\} \subseteq W \subseteq [0, 1]$. *Any function* v *from* V *to* W *is called valuation of variables which naturally extends to the evaluation of formulas in the following way:*

$$v(0) = 0, v(1) = 1,$$
$$v(\phi \& \psi) = v(\phi) * v(\psi)$$
$$v(\phi \rightarrow \psi) = v(\phi) \Rightarrow v(\psi)$$

where $*$ *and* $\Rightarrow$ *are the truth functions of* $\&$ *and* $\rightarrow$ *respectively. In our case* $*$ *will be some t-norm and* $\Rightarrow$ *will be its residua.*
Propositional Logic can be extended by quantifiers in different ways. We will

be interested in extension of propositional Logic by propositional quantifiers. In contrast to classical propositional Logic, propositional quantification may increase expressivity of Many valued propositional Logic. In classical propositional logic, the supremum and infimum of truth functions is expressed by propositional quantification. This correspondence can be extended to Many valued logics by using fuzzy quantifiers.
We will add quantifiers $\forall x$ *and* $\exists x$ *to the language. Free and bounded variables are defined in the usual way. Evaluation of formulas* $\forall x\phi(x)$ *and and* $\exists x\phi(x)$ *is defined as follows:*

$$v(\forall x\phi(x)) = inf\{w(\phi(x))|w \sim_x v\}$$
$$v(\exists x\phi(x)) = sup\{w(\phi(x))|w \sim_x v\}$$

Where $w \sim_x v$ *means that evaluations* v *and* w *agree on all variables except possible* x, *i.e.* $w \sim_x v$ *iff* $(w(x) \neq v(x)) \wedge (if\ y \neq x\ then\ w(y) = v(y))$
It is obvious that for different t-norms corresponding propositional quantifiers are different. One can prove that there is uncountable number of different quantified Gödel Logics.

Remark 4.1.1. It is easy to check that in contrast to classical and Łukasiewicz propositional Logic, in quantified propositional Gödel Logic $\forall x\phi \leftrightarrow \neg(\exists x(\neg\phi))$ is not valid formula.

4.2 Quantifier Elimination and interpolation

Definition 4.2.1. Logic L admits quantifier elimination if for all formulas ϕ of this logic we can find a quantifier free formula ψ in the same language which contains only free variables of ϕ and $L \vdash \phi \leftrightarrow \psi$.

As we are interested in quantified propositional many valued logics, we will give some examples of such logics which allows propositional quantifier elimination.

Example 4.2.2. Quantified propositional Gödel Logic on [0,1] which is denoted by QGL. Language of QGL is same as in previous subsection. Following are axioms of QGL:
1. $X \to (Y \to X)$
2. $(X \to (Y \to Z)) \to ((X \to Y) \to (X \to Z))$
3. $X \wedge Y \to X$
4. $X \wedge Y \to Y$
5. $X \to (Y \to (X \wedge Y))$
6. $X \to (X \vee Y)$

7. $Y \rightarrow (X \vee Y)$
8. $((X \rightarrow Z) \wedge (Y \rightarrow Z)) \rightarrow ((X \vee Y) \rightarrow Z)$
9. $X \wedge (X \rightarrow 0)) \rightarrow Y$
10.$(X \rightarrow (X \rightarrow 0)) \rightarrow (X \rightarrow 0)$
11.$0 \rightarrow X$
12.$X \rightarrow 1$
13.$(X \rightarrow Y) \vee (Y \rightarrow X)$
14.$X(A) \rightarrow \exists x X(x)$
15.$\forall x X(x) \rightarrow X(A)$
16.$\forall x(A^{(x)} \vee B) \rightarrow A^{(x)} \vee \forall x B$
17.$\forall x(A^{(x)} \rightarrow x \vee x \rightarrow B^{(x)}) \rightarrow (A^{(x)} \rightarrow B^{(x)})$
Where $A^{(x)}$ denotes that x does not occur free in A.
Deduction rules are:

$$\frac{X \quad X\rightarrow Y}{Y}\text{MP} \qquad \frac{Z(a)\rightarrow Y^{(a)}}{\exists x Z(x)\rightarrow Y^{(a)}}R\exists \qquad \frac{Y^{(a)}\rightarrow Z(a)}{Y^{(a)}\rightarrow \forall x Z(x)}R\forall$$

First 12 axioms are axioms of Intuitionistic logic, 13 is linearity axiom, 17 is density axiom. For any formula ϕ of this logic there exists quantifier-free ψ such that $\vdash_{QGL} \phi \leftrightarrow \psi$. (see proof in [5])

Example 4.2.3. It is easy to see that to have a quantifier elimination in a finite-valued logic (where minimum and maximum are definable) it is enough to add to the language the finite number of constants, one constant for each truth value. For example if $\{a_0, ...a_n\}$ is a set of truth values and we added $\{a'_0, ...a'_n\}$ constants to the language, s.t. for each evaluation v and for all $i \in \{0, 1, ..., n\}$ $v(a'_i) = a_i$, then obviously $\forall x(F(x, \bar{y})) \leftrightarrow F(a_0, \bar{y}) \wedge F(a_1, \bar{y}) \wedge ... \wedge F(a_n, \bar{y})$ and $\exists x(F(x, \bar{y})) \leftrightarrow F(a_0, \bar{y}) \vee F(a_1, \bar{y}) \vee ... \vee F(a_n, \bar{y})$ ()

Now we will provide definitions of interpolation and uniform interpolation and easy observations about relations between uniform interpolation, interpolations and quantifier elimination.

Definition 4.2.4. Logic L admits interpolation if for all A and B s.t. $\vdash_L A \rightarrow B$, there exists a formula $I(A, B)$ which contains only common variables of A and B such that $\vdash_L A \rightarrow I(A, B) \rightarrow B$.

Definition 4.2.5. Logic L admits uniform interpolation if for all A and B s.t. $\vdash_L A \rightarrow B$, there exists formulas $I(A, V)$ and $J(B, V)$ where V is a set of common variables of A and B, such that $\vdash_L A \rightarrow I(A, B) \rightarrow$J(B,V)$\rightarrow B$.

Remark 4.2.6. Obviously if L admits uniform interpolation, then it admits interpolation as well.

Remark 4.2.7. Interpolation is strongly connected with the existence of analytic properties of the deduction system. Let us consider a Hilbert style calculus for the logic, where the rule

$$\frac{A \to B \quad B \to C}{A \to C}$$

is derivable. Then interpolation implies that there are tree-like proofs of $A \to B$ and $B \to C$ which involve only variables which are respectively in A and C.

Remark 4.2.8. Let us analyze the truth functions of propositional quantifiers, it is clear that following sentences are tautologies: $A(x, y) \to \exists x A(x, y)$, $\forall x A(x, y) \to A(x, y)$. As usual this tautologies are axioms of the logic with quantifiers, and we will assume that we are considering logic like this, so we assume that this tautologies are also provable. It is also obvious that if $Z(a) \to Y^{(a)}$ and $Y^{(a)} \to Z(a)$ is a tautology then $\exists x Z(x) \to Y^{(a)}$ and $Y^{(a)} \to \forall x Z(x)$ respectively are as well tautologies, and as usual this relations in logic with quantifiers are derivation rules, so we will assume that if $\vdash Z(a) \to Y^{(a)}$ then $\vdash \exists x Z(x) \to Y^{(a)}$ and if $\vdash Y^{(a)} \to Z(a)$ then $\vdash Y^{(a)} \to \forall x Z(x)$

Let us consider $\vdash_L A \to B$, If $V(A) - V(B) \neq \emptyset$ ($V(A)$ is a set of variables of formula A) then $\vdash A \to B$ is of the form $\vdash_L A(\bar{x}, \bar{y}) \to B(\bar{y}, \bar{z})$, from the previous remark it is clear that $\vdash_L A(\bar{x}, \bar{y}) \to \exists \bar{x} A(\bar{x}, \bar{y})$ and $\vdash_L \exists \bar{x} A(\bar{x}, \bar{y})) \to B(\bar{y}, \bar{z})$. If we have quantifier elimination in L and $\vdash \exists \bar{x} A(\bar{x}, \bar{y}) \leftrightarrow A'(\bar{y})$ then it is clear that $A'(\bar{y})$ is the interpolant of $\vdash_L A(\bar{x}, \bar{y}) \to B(\bar{y}, \bar{z})$.
Now if $V(A) - V(B) = \emptyset$ then $\vdash A \to B$ is of the form $\vdash_L A(\bar{x}) \to B(\bar{x}, \bar{y})$, from previous remark it is clear that $\vdash_L \forall \bar{y} B(\bar{x}, \bar{y}) \to B(\bar{x}, \bar{y})$ and $\vdash_L A(\bar{x}) \to \forall \bar{y} B(\bar{x}, \bar{y})$. If we have quantifier elimination in L and $\vdash \exists \bar{y} B(\bar{x}, \bar{y}) \leftrightarrow B'(\bar{x})$ then it is clear that $B'(\bar{x})$ is the interpolant of $\vdash_L A(\bar{x}) \to B(\bar{x}, \bar{y})$.
So from this we can conclude the following:

Lemma 4.2.9. *If a logic extended by propositional quantifiers admits propositional quantifier elimination, then it has uniform interpolation.*
◀ *Take* $\forall y_1 ... y_i A$ *for* $I(A, V)$ *and* $\exists z_1 ... z_j B$ *for* $J(B, V)$ *where* $\{y_1, ... y_i\} = V(A) - V$ *and* $\{z_1 ... z_j\} = V(B) - V$ ▶
and as a corollary of this lemma we have :

Corollary 4.2.10. *If a logic L has no uniform interpolation, then L extended by propositional quantifiers does not admit propositional quantifier elimination.*

Corollary 4.2.11. *If a logic L has no interpolation, then L extended by propositional quantifiers does not admit propositional quantifier elimination.*

Now we will give theorem without proof which states that Łukasiewicz Logic Ł does not admit interpolation.

Theorem 4.2.12. *There is a formula Ψ, such that*
1.Ψ is a tautology in every triangular logic.
2.Ψ interpolates in G.
3.Ψ does not interpolate in Ł.
4.Ψ does not interpolate in Π.
(Ψ interpolates means that Ψ is of the form $\phi_1 \rightarrow \phi_2$)
◀ *see proof of Theorem 6 in [2]* ▶

Here is the formula Ψ from previous theorem:
$\Psi = min(max(x,p) \rightarrow p,\ max(x,p)) \longrightarrow max(max(y,p) \rightarrow p,\ max(y,p))$

So as we can see from previous theorem Ψ is tautology in Ł and because of complexity of L it is also provable, and it does not interpolate in Ł, it means we have no interpolation in Ł and by corollary 5.2.7 we can not eliminate propositional quantifiers in Ł extended by propositional quantifiers.

Now we will give theorem from [2] without proof.

Theorem 4.2.13. *No extension of Ł by a finite number of division operators has interpolation.*
◀ *see proof of corollary 3 in [2]* ▶

As a corollary of previous theorem we have following:

Corollary 4.2.14. *Let $QP\text{Ł}$ be quantified propositional Łukasiewicz Logic, then no extension of $QP\text{Ł}$ by a finite number of division operators admits quantifier elimination.*

Chapter 5

Quantifier Elimination in Quantified Propositional Łukasiewicz Logic

In this section we will extend propositional Łukasiewicz logic by all division operators and propositional quantifiers and will prove that we can eliminate quantifiers.

5.1 Quantified Propositional Łukasiewicz Logic by all Division Operators

Quantified Propositional Łukasievicz Logic extended by all Division Operators we will denote by QPŁ+$\square$. Language of QPŁ+$\square$ contains:
1. Infinite set of variable $V = \{x, x_1, x_2 ..., y, y_1, y_2, ...\}$
2. Constants 0 and 1
3. Infinite number of connectives $\&, \rightarrow, \square_1, \square_2, \square_3, ...$
4. Propositional quantifiers $\forall$, $\exists$.
Formulas, substitutions and free and bounded variables are defined in the usual way.

As a truth value set we will take [0,1]. Any function v from V to [0,1] is called a valuation of variables which naturally extends to the evaluation of formulas in the following way:

$$v(0) = 0, v(1) = 1,$$

$$v(\phi\&\psi) = v(\phi) * v(\psi)$$
$$v(\phi \to \psi) = v(\phi) \Rightarrow v(\psi)$$
$$\textit{for all } i \in \{1,2,3,...\}\ v(\Box_i\phi) = \blacksquare_i v(\phi)$$
$$v(\forall x\phi(x)) = inf\{w(\phi(x))|w \sim_x v\}$$
$$v(\exists x\phi(x)) = sup\{w(\phi(x))|w \sim_x v\}$$

Where $*$ is truth function of Łukasiewicz conjunction ($x * y = max\{x + y - 1, 0\}$), $\Rightarrow$ is truth function of Łukasiewicz implication ($x \Rightarrow y = min\{1 - x+y, 1\}$) and finally $\blacksquare_i$-s are Łukasziewic division operators ($\blacksquare_i x = \frac{x}{i} + \frac{i-1}{i}$).

Definition 5.1.1. A Formula ϕ is 1-tautology of QPŁ+□ iff $v(\phi) = 1$ for every evaluation v.

Following are axioms of QPŁ+□ :
$(A1)$ $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$ transitivity of implication
$(A2)$ $(\phi\&\psi) \to \phi$ conjunction implies first conjunct
$(A3)$ $(\phi\&\psi) \to (\psi\&\phi)$ commutativity of conjunction
$(A4)$ $(\phi \wedge \psi) \to (\psi \wedge \phi)$ commutativity of $\wedge$
$(A5)$ $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi\&\psi) \to \chi)$
$(A6)$ $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$ variant of proof by cases
$(A7)$ $\overline{0} \to \phi$. $\overline{0}$ implies everything.
For all $i \in \{1, 2, 3...\}$ we have following Axioms:
$(A8_i)(\Box_i\phi)^i \leftrightarrow \phi$.
$(A9_i)(\psi^i \to \phi) \to (\psi \to \Box_i\phi)$
$(\neg\neg)$ $\neg\neg\phi \to \phi$
$(\exists 1)$. $X(A) \to \exists x X(x)$
$(\forall 1)$.$\forall x X(x) \to X(A)$
$(\forall\vee)$.$\forall x(A^{(x)} \vee B) \to A^{(x)} \vee \forall x B$
Where $\phi \wedge \psi$ is $\phi\&(\phi \to \psi)$ and $\phi \vee \psi$ stands for $((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$.
First 10 axioms are axioms of Ł+□, rest axioms are axioms for quantifiers. Derivation rules are same as in QGL i.e.

$$\frac{X \quad X \to Y}{Y}\text{MP} \qquad \frac{Z(a) \to Y^{(a)}}{\exists x Z(x) \to Y^{(a)}}R\exists \qquad \frac{Y^{(a)} \to Z(a)}{Y^{(a)} \to \forall x Z(x)}R\forall$$

Remark 5.1.2. Obviously QPŁ+□ is not finite axiomatizable

5.2 New connectives and McNaughton Theorem

In this subsection we will try to represent formulas of QPŁ+□ in more appropriate way for our aims. Then we will provide McNaughton Theorem

and one for us important generalization of it without proof.
Let us define two new connectives in our Logic:
1. $X \veebar Y$ is $\neg X \to Y$ ($\neg X$ stands for $X \to 0$)
2. $X \barwedge Y$ is $X \& \neg Y$
It is easy to check that truth function of $\veebar$ is $\dot{+}$ and truth function of $\barwedge$ is $\dot{-}$ where $x \dot{+} y = min\{1, x+y\}$ and $x \dot{-} y = max\{0, x-y\}$.
From this moment we will not distinguish syntactic connectives and truth functions i.e. we treat formulas as polynomials. In not all cases we will use notations of connectives, sometimes we will use notations of truth functions.
Following Lemma gives us opportunity to change $\Box_i$ in our language by $\Box_i^*$ where $\Box_i^* x = \frac{x}{i}$

Lemma 5.2.1. *$\Box_i$ and $\Box_i^*$ are mutually expressible in QPŁ+$\Box$.*
◄. *Since $\Box_i x = \frac{x}{i} + \frac{i-1}{i}$, we have $\Box_i 0 = \frac{i-1}{i}$ and $\Box_i^* x = \Box_i x \dot{-} \Box_i 0$.*
On the other hand $\Box_i x = \Box_i^* x \dot{+} \underbrace{\Box_i^* 1 \dot{+} ... \dot{+} \Box_i^* 1}_{i-1 \ times}$ ►

From this moment, by misuse of notation, we shall always tacitly use $\Box_i$ instead of $\Box_i^*$ without further notice.

Remark 5.2.2. For all $k \in \{1, 2, 3, ...\}$ by kx we will denote $\underbrace{x \dot{+} ... \dot{+} x}_{k \ times}$.

Remark 5.2.3. It is clear that we can express in Ł+$\Box$ all rational numbers in [0,1], even more for all rational numbers in $[0, 1]$ we can define product on this number in the following way $\frac{p}{q} \bullet x = p\Box_q x$

Remark 5.2.4. It is easy to check that $x \& y = ((\frac{x}{2} \dot{+} \frac{y}{2}) \dot{-} \frac{1}{2}) \dot{+} ((\frac{x}{2} \dot{+} \frac{y}{2}) \dot{-} \frac{1}{2}) = ((\Box_2 x \dot{+} \Box_2 y) \dot{-} \Box_2 1) \dot{+} ((\Box_2 x \dot{+} \Box_2 y) \dot{-} \Box_2 1)$ and $x \to y = 1 \dot{-} (x \dot{-} y)$. So it means that using only $\dot{-}, \dot{+}, \Box_1, \Box_2, \Box_3 ...$ we can express every formula of Ł+$\Box$.

From previous remarks it is intuitively clear that truth functions of Ł+$\Box$ are piecewise linear functions on [0,1]. Now we will give McNaughton theorem and easy generalization of it without proofs.

Theorem 5.2.5. *(McNaughton). The truth functions of Łukasiewicz logic coincide with the continuous piecewise linear functions with integer coefficients on [0,1].*
◄ *see proof in [6]* ►

Theorem 5.2.6. *Let f be a piecewise linear function on [0,1], then f is continuous with rational coefficients iff f is definable Ł+□*
◄ *see proof of Lemma 9 in [2]* ►

5.3 Quantifier elimination

As we have seen in previous subsection truth functions of Ł+□ are continuous piecewise linear functions with rational coefficients. Our aim is to take a formula A of QPŁ+□ with quantifiers and find provable equivalent formula A' which is formula of Ł+□and contains only free variables of A. In this subsection we will show that for any formula $A(x, \bar{y})$ of Ł+□ there is a formula $A'(\bar{y})$ of Ł+□ such that $\exists x A(x, \bar{y}) \leftrightarrow A'(\bar{y})$ is a tautology.
Assume $A(x, \bar{y})$ is given, for all fixed $\bar{y'}$ we will find maximum of $A(x, \bar{y'})$on [0,1]. So it means that we want to find a maximum of truth function of $A(x, \bar{y'})$ which is a continuous piecewise linear function with rational coefficients. From elementary Mathematical Analyzes course and from Figure 1 it is obvious that the maximum can be reached either in so called minimax points or in 0 or in 1.

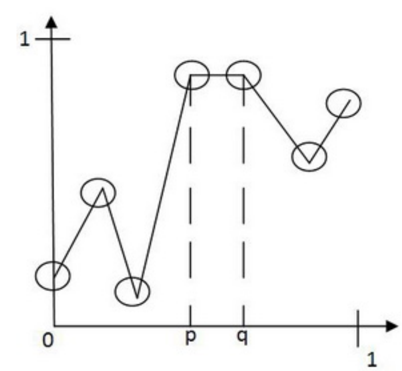

Figure 5.1: continuous piecewise linear function and minimax points

So if $\{a_1, a_2, ...a_n\}$ are a minimax points of $A(x, \bar{y'})$ then it is obvious that $max\{A(x, \bar{y'})|x \in [0, 1]\} = max\{A(x, \bar{y'})|x \in \{a_1, a_2, ...a_n\} \cup \{0, 1\} = A(a_1, \bar{y'}) \vee A(a_2, \bar{y'}) \vee ...A(a_n, \bar{y'}) \vee A(0, \bar{y'}) \vee A(1, \bar{y'})$. So if we can find all minimax points then it means that the maximum from function meanings in this points is the maximum of function on (0,1). Note that in our case set of all minimax points are finite.

Let us define strictly minimax point of function.

Definition 5.3.1. Let $f(x)$ be a truth function of Ł+□ formula with one variable then $t \in [0,1]$ is a minimax point of f if $t \in \{0,1\}$ or $f(x)$ is not differentiable in t .

Definition 5.3.2. Let $f(x,\bar{y})$ be a truth function of Ł+□ formula then for all fixed $\bar{y}$ we will define a minimax point of $f(x,\bar{y})$ as minimax point of $f'(x) = f(x,\bar(y))$

Remark 5.3.3. Our definition of minimax points differs from local extremum points but in case of truth functions of formulas of Ł+□ set of meanings of function in minimax points and a set of meanings of function in local extremum points are same i.e. $\{f(t)|t \in MM(f)\} = \{f(t)|t \in LE(f)\}$ where f is a truth function of a formula of Ł+□ and $MM(f)$ and $LE(f)$ respectively are sets of minimax and local extremum points. On Figure 1 all points in circles are minimax points, but besides of minimax points every point between p and q are local extremum points. So local extremum points can be not finite when minimax points are always finite.

Lemma 5.3.4. *Denote by $MM(f)$ set of minimax points of f function. If f_1 and f_2 are truth functions of formulas of Ł+□ then $MM(f_1+f_2) \subseteq MM(f_1) \cup MM(f_2)$*

◀ *Assume $t \in MM(f_1+f_2)$ and $\neg(t \in MM(f_1) \cup MM(f_2))$, because of $\neg(t \in MM(f_1) \cup MM(f_2))$ we have $\neg(t \in \{0,1\})$ and both of f_1 and f_2 are differentiable in t and it means that f_1+f_2 is also differentiable in t, contradiction.* ▶

Lemma 5.3.5. *If f_1 and f_2 are the truth functions of formulas of Ł+□ then $MM(f_1-f_2) \subseteq MM(f_1) \cup MM(f_2)$*
◀ *Exactly as in the proof of previous lemma* ▶

Remark 5.3.6. Generally $MM(f_1+f_2) = MM(f_1) \cup MM(f_2)$ is not true. On Figure 2 it is easy to check that the sum of red and black functions is constant and is equal to 1. So minimax points are only 0 and 1, but not 1/2.

Remark 5.3.7. It is obvious that for all i $MM(f) = MM(\Box_i f)$

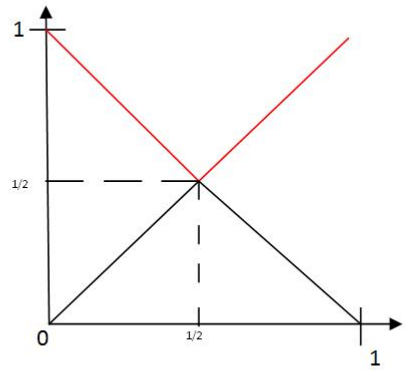

Figure 5.2: counterexample of $MM(f_1 + f_2) = MM(f_1) \cup MM(f_2)$

Now question is what will be $MM(f)$ if $f = f_1 \dot{+} f_2$? It is clear that if $f_1 + f_2 \leqslant 1$ then $MM(f) \subseteq MM(f_1) \cup MM(f_2)$ but it is also clear that if $f_1 + f_2 > 1$ then we have new minimax points, different from minimax points of f_1 and f_2. It is obvious that this new minimax points are points where $f_1 + f_2 = 1$, but not all of them. For example consider $f(x) = x\ if\ x < \frac{1}{2}$ and $f(x) = \frac{1}{2}\ if\ x \geqslant \frac{1}{2}$ and $g = f \dot{+} f$, then minimax points of g are 0,1 and $\frac{1}{2}$. For all $x \in (\frac{1}{2}, 1)$ $g(x) = 1$ but none of them is minimax point of g. Thing is if on whole interval $f_1 + f_2$ is 1 we can omit (do not include in set of minimax points) this interval, because none of them will be a minimax point(since inside of interval $f_1 + f_2$ is constantly 1 and it is differentiable), only borders of this interval can be the minimax points.

Definition 5.3.8. By $P(f, g)$ we will denote set of such x-s where $f(x) \dot{+} g(x) = 1$ and for all $\epsilon > 0$ there exists y such that $x - \epsilon < y < x + \epsilon$ and $f(y) \dot{+} g(y) \neq 1$ i.e.
$P(f, g) = \{x | (f(x) \dot{+} f(y) = 1)\ and\ (\forall \epsilon > 0, \exists y \in (x - \epsilon, x + \epsilon)\ s.t.\ (f(y) \dot{+} g(y) \neq 1))\}$

Lemma 5.3.9. *Let f_1 and f_2 be piecewise linear functions then $MM(f_1 \dot{+} f_2) \subseteq MM(f_1) \cup MM(f_2) \cup P(f_1, f_2)$*
◀ *Assume $x \in MM(f_1 \dot{+} f_2)$. If $(f_1 + f_2)(x) < 1$ then $x \in MM(f_1 + f_2)$ so $x \in MM(f_1) \cup MM(f_2) \subseteq MM(f_1) \cup MM(f_2) \cup P(f_1, f_2)$. If $(f_1 + f_2)(y) > 1$ then since f is continuous there exists open interval $I = (a, b)$ such that $y \in I$ and for all $z \in I$ $(f_1 + f_2)(z) > 1$, so $(f_1 \dot{+} f_2)(z) = 1$ and it means that $f_1 \dot{+} f_2$ is differentiable in y and y can not be a minimax point. So minimax points different from minimax points of f_1 and f_2 can be only*

x-s in which $(f_1 + f_2)$ is 1. Consider now $t \in \{x|(f_1 + f_2)(x) = 1\}$. $\forall\epsilon > 0, \exists y \in (t - \epsilon, t + \epsilon)$ s.t. $(f_1(y) \dot{+} f_2(y) \neq 1)$ does not hold means that there exists $\epsilon > 0$ s.t. $\forall y \in (t - \epsilon, t + \epsilon)$ $f_1(y) \dot{+} f_2(y) = 1$ and it means that $f_1 \dot{+} f_2$ is differentiable in t. So we have: if $(f_1 + f_2)(t) = 1$ and $\neg(t \in P(f_1, f_2))$ then t can not be a minimax point. This completes the proof. ►

Obviously case when $f = f_1 \dot{-} f_2$ is completely same as case $f = f_1 \dot{+} f_2$

Definition 5.3.10. By $M(f, g)$ we will denote set of such x-s where $f(x) \dot{-} g(x) = 0$ and for all $\epsilon > 0$ there exists y such that $x - \epsilon < y < x + \epsilon$ and $f(y) \dot{-} g(y) \neq 0$ i.e.
$M(f, g) = \{x|(f(x) \dot{-} f(y) = 0)\ and\ (\forall\epsilon > 0, \exists y \in (x - \epsilon, x + \epsilon)\ s.t.\ (f(y) \dot{-} g(y) \neq 0))\}$

Lemma 5.3.11. *Let f_1 and f_2 be piecewise linear functions then $MM(f_1 \dot{-} f_2) \subseteq MM(f_1) \cup MM(f_2) \cup M(f_1, f_2)$*
◄ *Analogous to the proof of previous Lemma* ►

Remark 5.3.12. Obviously for all piecewise linear functions f and g $P(f, g)$ and $M(f, g)$ are finite sets.

Now the question is how can we find elements of $P(f, g)$ and $M(f, g)$? We are going to find finite sets in which $P(f, g)$ and $M(f, g)$ are included. Obviously for all $x \in [0, 1]$ and f and g $f(x) \dot{+} g(x)$ is either 1 or $f(x) + g(x)$ and $f(x) \dot{-} g(x)$ is either 0 or $f(x) - g(x)$. As we have seen all formulas of Ł+□ can expressed in terms of $\dot{+}, \dot{-}, \Box_1, \Box_2, ...$ Now we will give inductive definition of the set of possible linear meanings of formula which we will denote by $PL(f)$

Definition 5.3.13. $PL(f) = \{f\}$ if f is either variable or constant
$PL(\Box_i f) = \{\Box_i u | u \in PL(f)\}$
$PL(f \dot{+} g) = \{1\} \cup \bigcup_{u \in PL(f), v \in PL(g)} (u + v)$
$PL(f \dot{-} g) = \{0\} \cup \bigcup_{u \in PL(f), v \in PL(g)} (u - v)$

Remark 5.3.14. For all function f elements of $PL(f)$ are linear functions.

Example 5.3.15. $PL((x\dot{+}y)\dot{-}\Box_i z) = \{0\} \cup \bigcup_{u \in PL(x\dot{+}y), v \in PL(\Box_i z)} (u - v) = \{0\} \cup \bigcup_{u \in PL(x\dot{+}y)} (u - \Box_i z) = \{0, 1 - \Box_i z, (x+y) - \Box_i z\}$

Cardinality of $PL(f)$ is always finite and depends on number of $\dot{+}$ and $\dot{-}$.

Lemma 5.3.16. *For all f and $\bar{x} \in [0,1]^m$ ($\bar{x} = (x_1 ... x_m)$) there is $f^x \in PL(f)$ such that $f(\bar{x}) = f^x(\bar{x})$*
◀ *Proof by induction on numbers of $\dot{+}$ and $\dot{-}$ in f. Without loss of generality we can assume that $m = 1$.If there are no $\dot{+}$ and $\dot{-}$ in f it is clear. Now assume that for $k < n$ if k is number of $\dot{+}$ and $\dot{-}$ in f then for all $x \in [0,1]$ there is $f^x \in PL(f)$ such that $f(x) = f^x(x)$ and prove for f with n $\dot{+}$ and $\dot{-}$ in it.*
1.$f = f_1 \dot{+} f_2$. Let us take any $x \in [0,1]$ if $f_1(x) + f_2(x) > 1$ then $f(x) = 1$ and off course $1 \in PL(f_1 \dot{+} f_2)$. Now if $f_1(x) + f_2(x) \leqslant 1$ then by induction premise for all $x \in [0,1]$ there exists f_1^x and f_2^x s.t. $f_1^x \in PL(f_1)$, $f_2^x \in PL(f_2)$ and $f_1(x) = f_1^x(x)$, $f_2(x) = f_2^x(x)$. By definition of $PL(f_1 \dot{+} f_2)$ it is obvious that $f^x = f_1^x + f_2^x \in PL(f_1 \dot{+} f_2) = PL(f)$ and $f(x) = f^x(x)$
2.$f = f_1 \dot{-} f_2$ is analogous to case 1.
3.$f = \Box_i f_1$. We can assume that $f_1 = \Box_{i_1}\Box_{i_2}...\Box_{i_n} g$ for some $i_1, i_2 ... i_n$ and $g = g_1 \circ g_2$ where $\circ \in \{\dot{+}, \dot{-}\}$or g does not contain connectives by 1. 2. and induction premise it is clear that for all $x \in [0,1]$ there is $g^x \in PL(g)$ such that $g(x) = g^x(x)$ and by the definition of $PL(\Box_i f)$ we have that if $f^x = \Box_i(\Box_{i_1}\Box_{i_2}...\Box_{i_n} g^x)$ then $f(x) = f^x(x)$. ▶

Definition 5.3.17. Let $f(x)$ be the function of one variable then:
1. Set $OP(f) = \{x \in [0,1] | \exists f_i \in (PL(f) - \{1\}) f_i(x) = 1\}$ we will call set of 1-points of f function.
2. Set $ZP(f) = \{x | \exists f_i \in (PL(f) - \{0\}) f_i(x) = 0\}$ we will call set of 0-points of f function.

Lemma 5.3.18. *1. $P(f,g) \subseteq OP(f \dot{+} g)$*
2.. $M(f,g) \subseteq ZP(f \dot{+} g)$
◀
1. Assume $x \in P(f,g)$, then $f(x) \dot{+} g(x) = 1$. By previous Lemma for all x there exist $f^x \in PL(f)$ and $g^x \in PL(g)$ s.t. $f^x(x) = f(x)$ and $g^x(x) = g(x)$. So $(f^x + g^x)(x) = 1$ and $f^x + g^x \in (PL(f \dot{+} g) - \{1\}$ it means that $x \in OP(f \dot{+} g)$
2. Is analogous to 1. ▶

Corollary 5.3.19. *Let f_1 and f_2 be piecewise linear functions then $MM(f_1 \dot{+} f_2) \subseteq MM(f_1) \cup MM(f_2) \cup OP(f_1 \dot{+} f_2)$*

Corollary 5.3.20. *Let f_1 and f_2 be piecewise linear functions then $MM(f_1 \dot{-} f_2) \subseteq MM(f_1) \cup MM(f_2) \cup ZP(f_1 \dot{-} f_2)$*

Division Operators give us opportunity to solve linear equations, and because of this we can construct $OP(f)$ and $ZP(f)$.
For simplicity we will consider formula with 2 variables. We want to solve for x in the equation $f(x,y) = 1$ where $f = h + g$, for this we will solve all equations $f_i(x,y) = 1$ where $f_i(x,y) \in PL(f(x,y))$, and then collect all solutions which are in [0,1] in one set. It is obvious from previous lemma that this will give us all solutions of $f(x,y) = 1$.
All elements of $PL(f(x,y))$ have one from following forms: $ax+by+c$, $ax+by-c$, $ax-by+c$, $ax-by-c$, $-ax+by+c$, $-ax-by+c$, $-ax+by-c$, $-ax-by-c$ where a, b and c are nonnegative rational numbers. So our equation can have the following form:$ax+by+c = 1$, $ax+by-c = 1$, $ax-by+c = 1$, $ax-by-c = 1$, $-ax+by+c = 1$, $-ax-by+c = 1$, $-ax+by-c = 1$, $-ax-by-c = 1$. Our aim is to write all solutions which are in [0,1] in terms of $\dot{+}$, $\dot{-}$ and $\square$. Solution has one from the following forms: $a'y+c'$, $a'y-c'$, $c'-a'y$, $-c'-a'y$ where a', c' are nonnegative rational numbers and $y \in [0,1]$. $\dot{+}$ and $\dot{-}$ are defined on $[0,1]$ and $a'y$ and c' can be outside of $[0,1]$, so we can not just replace $+$ and $-$ respectively with $\dot{+}$ and $\dot{-}$. To deal with this problem we will rewrite our solutions as follows: $a'y + c' = \underbrace{(\frac{a'y}{n} + \frac{c'}{n}) + ... + (\frac{a'y}{n} + \frac{c'}{n})}_{n\ times}$, $a'y - c' = \underbrace{(\frac{a'y}{n} - \frac{c'}{n}) + ... + (\frac{a'y}{n} - \frac{c'}{n})}_{n\ times}$, $c' - a'y = \underbrace{(\frac{c'}{n} - \frac{a'y}{n}) + ... + (\frac{c'}{n} - \frac{a'y}{n})}_{n\ times}$, $-c' - a'y = \underbrace{(\frac{-c'}{n} - \frac{a'y}{n}) + ... + (\frac{c'}{n} - \frac{a'y}{n})}_{n\ times}$ where n is the greatest integer s.t. $n \leqslant a' + c' + 1$. Now we will replace in our solutions all $+$ and $-$ respectively by $\dot{+}$ and $\dot{-}$, It is easy to check that if solution was in $[0,1]$ after replacing it will be again in $[0,1]$, if solution was negative after replacing it will be 0 and if solution was more than 1, after replacing it will be 1. Similarly when $f = h - g$ we will solve for x, $f(x,y) = 0$ for this we will do same procedure as above.
Obviously we can do same if number of variables differs from 2. Summariz-

ing all this we have following lemma:

Lemma 5.3.21. *Let $f(x,\bar{y})$ be a truth function of a Ł+□ formula, then for all $f_i \in PL(f)$ we can find a function $s(\bar{y})(s'(\bar{y}))$ of $\bar{y}$, which contains only $\dot{+}, \dot{-}, \square$ s.t. $f_i(s(\bar{y}),\bar{y}) = 1$ $(f_i(s'(\bar{y}),\bar{y}) = 0)$ and for all fixed $\bar{y}'$ and $x' \in [0,1]$ s.t $f_i(x',\bar{y'}) = 1$ $(f_i(x',\bar{y'}) = 0)$ holds $x' = s(y')$ $(x' = s'(y'))$*

So for all truth function $f(x,\bar{y})$ of Ł+□ formulas and for all fixed $\bar{y}$ we can find all $x \in [0,1]$ s.t. $f(x,\bar{y}) = 1$ and $f(x,\bar{y}) = 0$ and even more we can write it using only $\dot{+}$, $\dot{-}$ and □. If $\bar{y}$ is not fixed then we can find function depended on $\bar{y}$ which for all fixed $\bar{y}$ will give us a solution. For all fixed $\bar{y}$ we can consider $f^{\bar{y}}(x) = f(x,\bar{y})$ as a function of one variable. Obviously If we will collect all solutions (for x) of $f(x,\bar{y}) = 1$ $(f(x,\bar{y}) = 0)$ in one set this set will be $OP(f^{\bar{y}})$ ($ZP(f^{\bar{y}})$). For all fixed $\bar{y}$ $OP(f^{\bar{y}})$ is set of 1-points, but we need set of functions which for all fixed $\bar{y}$ will give set of 1-points. From the previous Lemma it is clear that we can construct this solutions.

Definition 5.3.22. Let $f(x,\bar{y})$ be a truth function of Ł+□ formula then:
1. Set $OS(f) = \{s(\bar{y}) | \exists f_i \in (PL(f) - \{1\}) f_i(s(\bar{y}),\bar{y}) = 1$ and s contains only $\dot{+}, \dot{-}$ and $\square\}$ we will call set of 1-solutions of f function.
2. Set $ZS(f) = \{s(\bar{y}) | \exists f_i \in (PL(f) - \{1\}) f_i(s(\bar{y}),\bar{y}) = 0$ and s contains only $\dot{+}, \dot{-}$ and $\square\}$ we will call set of 0-solutions of f function.

Lemma 5.3.23. *For all $f(x,\bar{y})$ and fixed $\bar{y'}$*
1. $OP(f') \subseteq \{s(\bar{y'}) | s \in OS(f)\}$ where $f'(x) = f(x,\bar{y'})$
2. $ZP(f') \subseteq \{s(\bar{y'}) | s \in ZS(f)\}$ where $f'(x) = f(x,\bar{y'})$
◀
1.Assume $x \in OP(f')$ it means that $\exists f'_i \in (PL(f') - \{1\}) f'_i(x) = 1\}$ but from the definition of $PL(f)$ it is clear that $f'_i \in PL(f')$ iff $\exists f_i \in PL(f)$ s.t. $f'_i(x) = f_i(x,\bar{y'})$, so $1 = f'_i(x) = f_i(x,\bar{y'})$. From Lemma 5.3.22 for all solutions of $f_i(x,\bar{y'}) = 1$ exists a function $s(\bar{y})$ which contains only $\dot{+}, \dot{-}$ and □ s.t $x = s(\bar{y'})$ and it means that $x \in \{s(\bar{y'}) | s \in OS(f)\}$
2.Analogous to 1. ▶

Definition 5.3.24. Let $f(x,\bar{y})$ be truth function of Ł+□ formula then define inductively set of functions of $\bar{y}$ $SMM(f)$:
$SMM(f) = \{0,1\}$ if f does not contain $\dot{+}$ and $\dot{-}$ (0, and 1 as constant

functions)
$SMM(\Box_i f) = SMM(f)$
if $f = h \dot{+} g$ then $SMM(f) = SMM(h) \cup SMM(g) \cup OS(f))$
if $f = h \dot{-} g$ then $SMM(f) = SMM(h) \cup SMM(g) \cup ZS(f))$

Lemma 5.3.25. *Let $f(x, \bar{y})$ be truth function of Ł+□ formula then for all fixed $\bar{y}$ $MM(f) \subseteq \{x|x = s(\bar{y}), s \in SMM(f)\}$*

◀ *Proof by induction on numbers of $\dot{+}$ and $\dot{-}$ in f. If f does not contain $\dot{+}$ and $\dot{-}$ then obviously only minimax points are 0 and 1 and from definition of $SSM(f)$ contains constant functions 0 and 1. Now assume that if numbers of $\dot{+}$ and $\dot{-}$ in f is $< n$ then $MM(f) \subseteq \{x|x = s(\bar{y}), s \in SMM(f)\}$ and prove for n.*
1.$f = f_1 \dot{+} f_2$. $SMM(f) = SMM(f_1) \cup SMM(f_2) \cup OS(f)$, by corollary 5.3.20
$MM(f_1 \dot{+} f_2) \subseteq MM(f_1) \cup MM(f_2) \cup OP(f_1 \dot{+} f_2)$ and by induction premise $MM(f_1) \subseteq \{x|x = s(\bar{y}), s \in SMM(f_1)\}$ and $MM(f_2) \subseteq \{x|x = s(\bar{y}), s \in SMM(f_2)\}$.
By Lemma 5.3.24 $OP(f_1 \dot{+} f_2) \subseteq \{x|x = s(\bar{y}), s \in OS(f_1 \dot{+} f_2)\}$, so we have
$MM(f_1 \dot{+} f_2) \subseteq MM(f_1) \cup MM(f_2) \cup OP(f_1 \dot{+} f_2) \subseteq$
$\{x|x = s(\bar{y}), s \in SMM(f_1)\} \cup \{x|x = s(\bar{y}), s \in SMM(f_2)\} \cup \{x|x = s(\bar{y}), s \in OS(f_1 \dot{+} f_2)\} =$
$\{x|x = s(\bar{y}), s \in (SMM(f_1) \cup SMM(f_1) \cup OS(f_1 \dot{+} f_2))\} =$
$\{x|x = s(\bar{y}), s \in SMM(f_1 \dot{+} f_2)\}$
2.$f = f_1 \dot{-} f_2$ is analogous to 1.
3.$f = \Box_i f_1$. We can assume that $f_1 = \Box_{i_1} \Box_{i_2} ... \Box_{i_n} g$ for some $i_1, i_2 ... i_n$ and $g = g_1 \circ g_2$ where $\circ \in \{\dot{+}, \dot{-}\}$or g does not contain connectives by 1. 2. and induction premise it is clear that $MM(g) \subseteq \{x|x = s(\bar{y}), s \in SMM(g)\}$ $f = \Box_i(\Box_{i_1} \Box_{i_2} ... \Box_{i_n} g)$, by remark 5.3.8 $MM(f) = MM(g)$ and by definition of SMM, $SMM(f) = SMM(g)$ and consequently $MM(f) \subseteq \{x|x = s(\bar{y}), s \in SMM(f)\}$
▶

Remark 5.3.26. Obviously for all f $Card(SMM(f))$ is finite.

Theorem 5.3.27. *Let $F(x, \bar{y})$ be a formula of Ł+□ and $f(x, \bar{y})$ a truth function of $F(x, \bar{y})$ then:*
1.$\exists x F(x, \bar{y}) \leftrightarrow F(s_1(\bar{y}), \bar{y}) \vee F(s_2(\bar{y}), \bar{y}) ... \vee F(s_n(\bar{y}), \bar{y})$ is 1-tautology in QPŁ+□

2.$\forall x F(x,\bar{y}) \leftrightarrow F(s_1(\bar{y}),\bar{y}) \wedge F(s_2(\bar{y}),\bar{y})... \wedge F(s_n(\bar{y}),\bar{y})$ *is 1-tautology in QPŁ+□*
where $\{s_1, s_2, ...s_n\} = SMM(f)$ *(We will not differ formula* s_i *from it is truth function* s_i*)*

◀

1. Let v *be an evaluation, then* $v(F(x,\bar{y})) = sup\{w(F(x,\bar{y}))|w \sim_x v\} = sup\{f(x, v(\bar{y}))|x \in [0,1]\} = max\{f(x, v(\bar{y}))|x \in MM(f)\}$ *by previous Lemma for* $v(\bar{y})$ *hold* $MM(f) \subseteq \{x|x = s(v(\bar{y})), s \in SMM(f)\}$ *but maximum of* $f'(x) = f(x, v(\bar{y}))$ *can be reached only in* $MM(f') = MM(f)$*, it means that* $max\{f'(x)|x \in MM(f)\} = max\{f'(x)|x \in A\}$ *for any* $A \supseteq MM(f)$*, so* $max\{f(x, v(\bar{y}))|x \in MM(f)\} = max\{f(x, v(\bar{y}))|x \in \{x|x = s(v(\bar{y})), s \in SMM(f)\} =$
$max\{f(s_i(v(\bar{y})), v(\bar{y}))|s_i \in SMM(f)\}$.
On the other hand $v(F(s_1(\bar{y}),\bar{y}) \vee F(s_n(\bar{y}),\bar{y})... \vee F(s_n(\bar{y}),\bar{y})) =$
$max(v(F(s_1(\bar{y}),\bar{y})), v(F(s_2(\bar{y}),\bar{y})), ..., v(F(s_n(\bar{y}),\bar{y}))) =$
$max(f(s_1(\bar{v}(y)), v(\bar{y})), f(s_2(\bar{v}(y)), \bar{v}(y)), ..., f(s_n(\bar{v}(y)), \bar{v}(y))) =$
$max\{f(s_i(v(\bar{y})), v(\bar{y}))|s_i \in SMM(f)\} = v(F(x,\bar{y}))$
2. Similar to 1.

▶

Now we want to prove that this equivalences is also provable in QPŁ+□. So we have $\exists x F(x,\bar{y}) \leftrightarrow F'(\bar{y}) = F(s_1(\bar{y}),\bar{y}) \vee F(s_2(\bar{y}),\bar{y})... \vee F(s_n(\bar{y}),\bar{y})$.
1.$F'(\bar{y}) \to \exists x F(x,\bar{y})$*. Because of* $(\exists 1)$ *axiom of QPŁ+□ for all* $i = 1, 2, ...n$ *we have* $F(s_i(\bar{y}),\bar{y}) \to \exists x F(x,\bar{y})$*. Let* $A \to C$ *and* $B \to C$ *be tautologies of Ł+□.* $A \to C$ *and* $B \to C$ *are tautologies in Ł+□ means that for all evaluation* v $v(A) \leqslant v(C)$ *and* $v(B) \leqslant v(C)$ *and it means that* $max\{v(A), v(B) \leqslant v(C)\}$ *thus* $A \vee B \to C$ *is also tautology of Ł+□ and because of completeness of Ł+□ we can prove it. So in Ł+□ following derivation rule is correct:*
$\frac{A \to C \quad B \to C}{A \vee B \to C}$
Obviously this rule also will be correct in QPŁ+□, and since for all $i = 1, 2, ...n$, $F(s_i(\bar{y}),\bar{y}) \to \exists x F(x,\bar{y})$*, we have* $F'(\bar{y}) = F(s_1(\bar{y}),\bar{y}) \vee F(s_2(\bar{y}),\bar{y})... \vee F(s_n(\bar{y}),\bar{y}) \to F(x,\bar{y})$ *is provable in QPŁ+□*
2. $\exists x F(x,\bar{y}) \to F'(\bar{y})$*. Again by* $(\exists 1)$ *axiom of QPŁ+□ we have that,* $F(x,\bar{y}) \to F(x,\bar{y})$ *is tautology, so for all evaluation* v $v(F(x,\bar{y})) \leqslant v(\exists x F(x,\bar{y}))$*.* $\exists x F(x,\bar{y}) \to F'(\bar{y})$ *is also a tautology and* $v(\exists x F(x,\bar{y})) \leqslant v(F'(\bar{y}))$*, so we have that* $v(F(x,\bar{y})) \leqslant v(F'(\bar{y}))$ *and it means that* $F(x,\bar{y}) \to F'(\bar{y})$ *is a tautology of Ł+□, so because of completeness of Ł+□ it is also provable and finally using rule* $R\exists$ *of QPŁ+□ we will get that* $\exists x F(x,\bar{y}) \to F'(\bar{y})$ *is also provable. In our case* $\exists$ *and* $\forall$ *are dual, so same is true for the universal*

quantifier, or we can prove it in the similar way. So we proved following:

Theorem 5.3.28. *(Quantifier Elimination) Let $F(x,\bar{y})$ be formula of Ł+□ and $f(x,\bar{y})$ truth function of $F(x,\bar{y})$ then:*
1. $\vdash_{QPL+\Box} \exists x F(x,\bar{y}) \leftrightarrow F(s_1(\bar{y}),\bar{y}) \vee F(s_2(\bar{y}),\bar{y}) ... \vee F(s_n(\bar{y}),\bar{y})$
2.$\vdash_{QPL+\Box} \forall x F(x,\bar{y}) \leftrightarrow F(s_1(\bar{y}),\bar{y}) \wedge F(s_2(\bar{y}),\bar{y}) ... \wedge F(s_n(\bar{y}),\bar{y})$
where $\{s_1, s_2, ... s_n\} = SMM(f)$ *(We will not differ formula s_i from its truth function s_i)*

This theorem naturally extends to the following:

Theorem 5.3.29. *(Quantifier Elimination) Let $F(x,\bar{y})$ be formula of QPŁ+□ then we can find quantifier free $F'(\bar{y})$ and $F''(\bar{y})$ s.t.*
1. $\vdash_{QGL+\Box} \exists x F(x,\bar{y}) \leftrightarrow F'(\bar{y})$
2.$\vdash_{QGL+\Box} \forall x F(x,\bar{y}) \leftrightarrow F''(\bar{y})$
◀ *If $F(x,\bar{y})$ is quantifier free than it follows from previous theorem. Without loss of generality we can assume that F has following form: $F(x,\bar{y}) = Q_1 z_1 Q_2 z_2, ... Q_n z_n G(z_1, z_2, ... z_n, x, \bar{y})$ where $Q_i \in \{\exists, \forall\}$, by previous theorem we can find $G'(z_1, z_2, ... z_{n-1}, x, \bar{y})) \leftrightarrow Q_n z_n G(z_1, z_2, ... z_n, x, \bar{y})$ so $F(x,\bar{y}) = Q_1 z_1 Q_2 z_2, ... Q_{n-1} z_{n-1} G'(z_1, z_2, ... z_{n-1}, x, \bar{y})$. Step by step can eliminate all quantifiers and finally we will get quantifier free formula which is equivalent to F.* ▶

As a corollary of previous theorem we have following corollaries:

Corollary 5.3.30. *QPŁ□ is decidable.*

Corollary 5.3.31. *QPŁ□ is recursively enumerable.*

From the quantifier elimination theorem and completeness of Ł+□ we have Completeness theorem of QPŁ□

Theorem 5.3.32. *QPŁ□ is complete i.e.* $\vdash_{QPL+\Box} F$ *iff* F *is 1-tautology .*

Conclusion

Let us summarize what have we achieved. We proved completeness of Basic Logic extended with all division operators with respect to BL■-algebras, then completeness of Łukasiewicz Logic extended with all division operators, and finally we proved that Łukasiewicz Logic extended by all division operators has quantifier elimination property, and as a corollary we obtained the completeness of this Logic.
Our algorithm of quantifier elimination is semantical and has some advantages, e.g. in contrast to usual syntactic methods it is not necessary to eliminate quantifiers of the same type one by one, because we can resolve linear equations not just for one variables but also for a vector of variables.
Let us discuss finally concept of a minimal extension. Observe that we can represent each natural number as a product of prime numbers, and that for all i and j $\Box_i(\Box_j x) = \Box_{ij} x$, *so we can express all division operators with only prime division operators and obviously the prime division operators set is minimal, in the sense that we can express all division operators by its members. Consequently the extension of Łukasiewicz Logic with prime division operators is the minimal extension of Łukasiewicz Logic which admits quantifier elimination. Clearly all above mentioned theorems holds also for this extension.*

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