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Hybrid Logic in the Calculus of Structures

by/von
Armin Troy

Supervisor/Betreuer:

Dr. Lutz Straßburger - INRIA Futurs and École Polytechnique

Overseeing Professor/Verantwortlicher Hochschullehrer:

Prof. Dr. Steffen Hölldobler - TU Dresden

Abstract

Hybrid logic is an extension of modal logic which allows to access the states of a Kripke structure directly from within the logic. This is achieved with nominals which are an additional kind of propositional symbols. Nominals can be used to identify states since they are true at exactly one state of the Kripke structure by definition. The calculus of structures is a type of inference system which does not only allow rule applications at the topmost connective of a formula, as it is the case for classical inference systems like sequent calculus, but also at subformula positions. An inference system in the calculus of structures is presented for a basic hybrid logic which contains the jump-operator @ as its only hybrid operator. A translation between this inference system and two different sequent calculae for hybrid logic is shown.

Contents

1 Introduction

Modal logic allows to reason about relational structures, i.e. sets of states on which several relations may be defined. Each of the modal operators, which compose a particular modal logic, depends on a corresponding relation in the underlying relational structure. The frame semantics for modal logic is based on the notion of a model, a relational structure combined with a valuation which defines at which states the propositional variables are mapped to true. Given such a model, the satisfaction relation tells whether some modal logic formula holds at a specific state or not. The definition of the satisfaction relation is the only place where the modal logic formulae are related to the actual states of the model and there is no way to access them from within the modal language, e.g. there is no possibility to access a particular state of a frame or to show the equivalence of different states in modal logic. Therefore frame properties like irreflexivity cannot be expressed. This motivates the use of hybrid logic, an extension of modal logic which allows to access the states directly. Hybrid logic uses nominals, an additional kind of propositional symbols which can be used to assign names to the states. In contrast to ordinary propositional variables, which can be true at any number of states, nominals are true at exactly one state of a relational structure. If the same nominal holds at two states, then we know that these two states must be equivalent. Similarly, it is possible to distinguish between different states within hybrid logic. The basic hybrid logic $\mathcal{H}(@)$ which is used throughout this thesis only contains one hybrid operator (an operator which depends on the presence of nominals), the satisfaction operator @. More expressive hybrid logics can be obtained by adding further hybrid operators.

In contrast to classical inference systems where applications of the inference rules can only deal with the top-most connective of a formula (when it is seen as a tree), systems which use the approach of deep inference allow rule applications at arbitrary sub-tree positions of the formula. The calculus of structures is a type of inference system, which implements the deep inference strategy and was introduced by Alessio Guglielmi [Gug08]. It was successfully applied to different logics, e.g. classical logic [Brü04] and linear logic [Str03]. This thesis deals with system $\mathsf{BH}{\downarrow}{\uparrow}$, a system in the calculus of structures for the hybrid logic $\mathcal{H}(@)$, which was proposed by Lutz Straßburger [Str07].

The structure of this thesis is as follows: in the remainder of section 1 the basic hybrid logic $\mathcal{H}(@)$ will be introduced formally and the calculus of structures will be presented for classical propositional logic. Afterwards, in section 2 Straßburger's system $\mathsf{BH}{\downarrow}{\uparrow}$ will be introduced followed by a discussion of its drawbacks and improvements to his system. In section 3, translations between $\mathsf{BH}{\downarrow}{\uparrow}$ and two different sequent calculae are presented. Although these translations do not show cut elimination as it is possible with similar translations for other logics, they show the admissibility of some other rules. The conclusion in section 4 points out alternative ways which might lead to cut elimination for $\mathsf{BH}{\downarrow}{\uparrow}$ and lists further open questions in the context of hybrid logic and deep inference.

1.1 The Basic Hybrid Logic $\mathcal{H}(@)$

Hybrid logic is an extension of modal logic which allows to access the states of a Kripke structure within the language itself. This is done by providing a way to label the states by means of an additional kind of atoms, the so-called nominals. A nominal is true at exactly one state of a Kripke structure, in contrast to propositional variables which may be true at various states. The set $\mathcal{V} = \{p, q, r, \cdots\}$ of propositional variables and the set of nominals $\mathcal{N} = \{i, j, k, \cdots\}$ are disjoint and the set of atoms $\mathcal{A}$ is defined to be the union $\mathcal{V} \cup \mathcal{N}$.

Definition 1.1. (syntax of $\mathcal{H}(@)$): The set of well-formed $\mathcal{H}(@)$ formulae is inductively defined to be the smallest set such that:

- $\top \in \mathcal{H}(@)$ and $\bot \in \mathcal{H}(@)$
- $p \in \mathcal{H}(@)$ for every $p \in \mathcal{V}$
- $i \in \mathcal{H}(@)$ for every $i \in \mathcal{N}$
- $\neg\varphi \in \mathcal{H}(@)$, $\Diamond\varphi \in \mathcal{H}(@)$, and $\Box\varphi \in \mathcal{H}(@)$ for every $\varphi \in \mathcal{H}(@)$
- $(\varphi \wedge \psi) \in \mathcal{H}(@)$ and $(\varphi \vee \psi) \in \mathcal{H}(@)$ for every $\varphi \in \mathcal{H}(@)$ and $\psi \in \mathcal{H}(@)$
- $@_i\varphi \in \mathcal{H}(@)$ for every $\varphi \in \mathcal{H}(@)$ and $i \in \mathcal{N}$ (satisfaction statement)

The units $\top$ and $\bot$ are not necessary for the definition of hybrid logic, but they are introduced here, since the calculus of structures which will be used in later sections includes units. The language $\mathcal{H}(@)$ uses @ as its only hybrid operator. In general, it is possible to define more hybrid operators, e.g the binder $\downarrow$, which lead to more expressive hybrid logics. The implication $(\varphi \supset \psi)$ is defined as an abbreviation for $(\neg\varphi \vee \psi)$. The connectives $\Box$ and $\Diamond$ are dual to each other, i.e. $\Box\varphi = \neg\Diamond\neg\varphi$ and $\Diamond\varphi = \neg\Box\neg\varphi$, and @ can be shown to be dual to itself, i.e. $@_i\varphi = \neg @_i \neg\varphi$. In the literature, @ is sometimes called *jump operator* or *at operator*.

In order to define the semantics of $\mathcal{H}(@)$, some further notions have to be introduced first. A *frame* $(\mathcal{W}, \rightarrow)$ is a set of states $\mathcal{W} = \{w, w', w'', \cdots\}$ equipped with a binary relation $\rightarrow \subseteq \mathcal{W} \times \mathcal{W}$. A *valuation* $v : \mathcal{A} \rightarrow \mathfrak{P}(\mathcal{W})$ is a mapping assigning to each atom the set of states at which the atom is true. For nominals $i \in \mathcal{N}$ these sets are restricted to singleton sets. A *model* $\mathcal{M} = (\mathcal{W}, \rightarrow, v)$ is a frame together with a valuation. Now the semantics of the hybrid language $\mathcal{H}(@)$ can be defined by the *Kripke satisfaction relation* in a similar way as for modal logic.

Definition 1.2. (semantics of $\mathcal{H}(@)$): Let $\mathcal{M} = (\mathcal{W}, \rightarrow, v)$ be a model and w a state. Then the Kripke satisfaction relation $\mathcal{M}, w \Vdash \varphi$ for $\mathcal{H}(@)$ is defined as follows:

$\mathcal{M}, w \Vdash \top$		holds for every $\mathcal{M}$ and w
$\mathcal{M}, w \Vdash \bot$		does not hold for any $\mathcal{M}$ and w
$\mathcal{M}, w \Vdash p$	iff	$p \in \mathcal{V}$ and $w \in v(p)$
$\mathcal{M}, w \Vdash \neg\varphi$	iff	$\mathcal{M}, w \nVdash \varphi$
$\mathcal{M}, w \Vdash (\varphi \wedge \psi)$	iff	$\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash (\varphi \vee \psi)$	iff	$\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$
$\mathcal{M}, w \Vdash \Diamond\varphi$	iff	$\exists w' \in \mathcal{W} : w \rightarrow w'$ and $\mathcal{M}, w' \Vdash \varphi$
$\mathcal{M}, w \Vdash \Box\varphi$	iff	$\forall w' \in \mathcal{W} : w \rightarrow w'$ implies $\mathcal{M}, w' \Vdash \varphi$
$\mathcal{M}, w \Vdash i$	iff	$i \in \mathcal{N}$ and $v(i) = \{w\}$
$\mathcal{M}, w \Vdash @_i\varphi$	iff	$\mathcal{M}, w' \Vdash \varphi$, where $v(i) = \{w'\}$

Based on the Kripke satisfaction relation we can now define the different notions of validity of $\mathcal{H}(@)$ formulae.

Definition 1.3. (validity of $\mathcal{H}(@)$ formulae): Let $(F) = (\mathcal{W}, \rightarrow)$ be a frame, $\mathcal{M} = (\mathcal{W}, \rightarrow, v)$ a model, and φ a formula. Then:

- φ is valid in the model $\mathcal{M}$, notation $\mathcal{M} \Vdash \varphi$, if $\mathcal{M}, w \Vdash \varphi$ for all $w \in \mathcal{W}$.
- φ is valid in the frame $\mathcal{F}$, notation $\mathcal{F} \Vdash \varphi$, if $(\mathcal{W}, \rightarrow, v) \Vdash \varphi$ for every valuation v.
- φ is valid, notation $\Vdash \varphi$, if $(\mathcal{W}, \rightarrow) \Vdash \varphi$ for every frame $(\mathcal{W}, \rightarrow)$.

If not stated explicitly, we are interested in the validity $\Vdash \varphi$ on arbitrary frames. The following two formulae are examples for $\mathcal{H}(@)$ formulae:

1. $(i \supset \neg\Diamond i)$
   This formula is only valid on irreflexive frames. It says that the state labelled with the nominal i may not have itself as its successor w.r.t. $\rightarrow$. By definition of validity on frames, all possible valuations have to be considered. Since i can denote any state of the frame in a particular valuation, this means that the frame must be irreflexive. This formula shows that hybrid logic is indeed an extension of modal logic, since irreflexivity cannot be expressed in modal logic.
2. $((@_i a \wedge @_i b) \supset @_i(a \wedge b))$
   The second formula shows the use of the jump operator. If we know that a holds at a state denoted by the nominal i and that b holds at the same state, we can conclude that $(a \wedge b)$ holds at this state as well.

1.2 Deep Inference and the Calculus of Structures

In classical inference systems like in sequent calculus, one usually only has inference rules to deal with the topmost connectives (viewing the formulae as trees). Systems which allow to apply inference rules at arbitrary depth within a formula are said to employ deep inference. One inference system which uses deep inference is the calculus of structures. This section introduces the calculus of structures for classical propositional logic and will be extended to $\mathcal{H}(@)$ in Section 2.

In the calculus of structures one works with syntactic objects called *structures*, which can be seen as intermediate between formulae and sequents. Formally, they are defined by the following grammar

$$R ::= a \mid \mathbf{t} \mid \mathbf{f} \mid \bar{R} \mid (R, R) \mid [R, R]$$

where a can be any atom. In this notation, negation is denoted by a bar, whereas (A, B) stands for the conjunction and $[A, B]$ for the disjunction of the structures A and B. Furthermore, the constants $\top$ and $\bot$ are denoted by the symbols $\mathbf{t}$ and $\mathbf{f}$ respectively. Capital letters $A, B, \ldots$ will be used for structures.

The equations from Figure 1 define a smallest congruence relation $=$ on the set of structures R. Each structure is an element of one equivalence class from the factor set $R/=$. That is, each structure is equivalent to all other structures from its equivalence class.

A normal form for structures is obtained by pushing down negations to the atoms by using the De Morgan laws and removing superfluous parentheses and units by associativity and the equivalence rules for units. Normal forms are not unique because of commutativity, but their number is finite for each equivalence class.

Before we can introduce inference rules in the calculus of structures, the notion of a *context* has to be introduced. Intuitively, a context can be seen as a structure which contains one occurrence of the hole $\{\}$. This can be formalised by the grammar

$$S ::= \{\} \mid (S, R) \mid [S, R]$$

where R stands for a structure. Contexts will be denoted by S, T, ... followed by the hole $\{\}$. A structure R can be plugged into a context $S\{\}$ by replacing the hole $\{\}$ with the structure R. The structure one obtains in this way is denoted by $S\{R\}$. By the definition of contexts the hole or structures which are plugged into the hole can never be within the scope of a negation.

Having introduced structures and contexts, we can now turn to the definition of inference rules and deductive systems.

Associativity

$$((A, B), C) = (A, (B, C))$$

$$[[A, B], C] = [A, [B, C]]$$

Commutativity

$$(A, B) = (B, A)$$

$$[A, B] = [B, A]$$

Units

$$[\mathbf{f}, A] = A$$

$$(\mathbf{t}, A) = A$$

Negation

$$\bar{\mathbf{f}} = \mathbf{t} \qquad \bar{\mathbf{t}} = \mathbf{f}$$

$$\overline{(A, B)} = [\bar{A}, \bar{B}] \qquad \overline{[A, B]} = (\bar{A}, \bar{B})$$

$$\bar{\bar{A}} = A$$

Context Closure

$$\text{if } A = B \text{ then} \qquad S\{A\} = S\{B\}$$

$$\bar{A} = \bar{B}$$

Figure 1: Equivalence relation = defined on structures.

Definition 1.4. An inference rules in the calculus of structures is of the form

$$\rho \frac{S\{T\}}{S\{R\}}$$

where $S\{\}$ is some context and R and T are schemes for structures. S can also be the empty context $\{\}$. The structure $S\{T\}$ is called the *premise* and $S\{R\}$ the *conclusion* of the rule. The inference rules can be seen as rewrite rules, where an instance of T is replaced by an instance of R. Furthermore, it is possible to impose additional constraints on the application of inference rules.

Definition 1.5. A *(deductive) system* $\mathcal{S}$ in the calculus of structures is a set of inference rules.

Definition 1.6. The *equivalence rule* $= \frac{T}{R}$ can be used to replace a structure by another one which is syntactically equivalent w.r.t. the equivalence relation =. The rule is implicitly included in every system and is sometimes dropped for obvious equivalences.

Figure 2 shows the inference rules of system SKSg for classical logic. See [Brü04] for an extensive discussion of classical logic in the calculus of structures.

(*identity*)	$i{\downarrow}\,\dfrac{S\{\mathbf{t}\}}{S\{[\bar{A},A]\}}$	$i{\uparrow}\,\dfrac{S\{(\bar{A},A)\}}{S\{\mathbf{f}\}}$	(*cut*)
(*switch*)	$s\,\dfrac{S\{(A,[B,C])\}}{S\{[(A,B),C]\}}$		
(*weakening*)	$w{\downarrow}\,\dfrac{S\{\mathbf{f}\}}{S\{A\}}$	$w{\uparrow}\,\dfrac{S\{A\}}{S\{\mathbf{t}\}}$	(*co-weakening*)
(*contraction*)	$c{\downarrow}\,\dfrac{S\{[A,A]\}}{S\{A\}}$	$c{\uparrow}\,\dfrac{S\{A\}}{S\{(A,A)\}}$	(*co-contraction*)

Figure 2: System SKSg.

In the calculus of structures we can try to find derivations which show that one structure is a logical consequence of another one, or we can show the validity of a structure by showing that it can be derived from the constant $\mathbf{t}$.

Definition 1.7. A *derivation* Δ is a finite chain of instances of inference rules from a deductive system $\mathcal{S}$:

$$\rho\,\dfrac{T}{\rho'\,\dfrac{T'}{\dfrac{\vdots}{\pi'\,\dfrac{}{\pi\,\dfrac{R'}{R}}}}}$$

and is denoted by $\Delta\left\Vert\begin{smallmatrix}T\\ \\ R\end{smallmatrix}\right.\mathcal{S}$. A single structure is also a derivation.

Definition 1.8. A derivation Δ can be put into some context $S\{\}$ to obtain the derivation $S\{\Delta\}$ as follows:

$$\Delta = \begin{array}{c} \rho\,\dfrac{T}{T'} \\ \rho'\,\dfrac{}{\vdots} \\ \pi'\,\dfrac{}{R'} \\ \pi\,\dfrac{}{R} \end{array} \quad\rightsquigarrow\quad S\{\Delta\} = \begin{array}{c} \rho\,\dfrac{S\{T\}}{S\{T'\}} \\ \rho'\,\dfrac{}{\vdots} \\ \pi'\,\dfrac{}{S\{R'\}} \\ \pi\,\dfrac{}{S\{R\}} \end{array}$$

Definition 1.9. A *proof* Π of R in some system $\mathcal{S}$ is a derivation starting with the unit $\mathbf{t}$ which only uses inference rules from $\mathcal{S}$. It is denoted by

$$\Pi \, \Big\| \, \mathcal{S} \atop R$$

Definition 1.10. The *dual rule* of some inference rule $\rho \frac{T}{R}$ corresponds to the principle of contraposition and is the rule obtained from ρ by replacing the premise with the negation of the conclusion and the conclusion with the negation of the premise, e.g.

$$i{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{[\bar{A}, A]\}} \quad \text{is dual to} \quad i{\uparrow}\,\frac{S\{(\bar{A}, A)\}}{S\{\mathbf{f}\}}$$

A system is called *symmetric* if for each inference rule the system also contains its dual rule. System SKSg is symmetric. For each rule $\rho{\downarrow}$, it contains its dual rule $\rho{\uparrow}$. Note that the rule s is dual to itself.

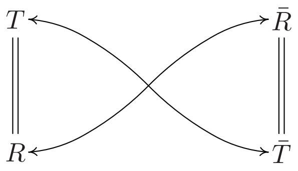


Figure 3: Symmetry of derivations in the calculus of structures.

In symmetric systems there is an interesting top-down symmetry for derivations (Figure 3). For a given derivation one obtains its *dual derivation* by reversing the order of the inference steps, by exchanging each structure by its negation and by replacing each inference rule by its dual rule, e.g.

$$w{\uparrow}\,\cfrac{[(a, \bar{b}), a]}{c{\downarrow}\,\cfrac{[a, a]}{a}} \quad \text{is dual to} \quad c{\uparrow}\,\cfrac{\bar{a}}{w{\downarrow}\,\cfrac{(\bar{a}, \bar{a})}{([\bar{a}, b], \bar{a})}}$$

Whereas the dual object of a derivation is again a derivation, the dual of a proof is a *refutation*, i.e. a derivation with $\mathbf{f}$ as its conclusion, e.g.

$$i{\downarrow}\,\cfrac{\mathbf{t}}{c{\downarrow}\,\cfrac{[(a, a), [\bar{a}, \bar{a}]]}{[(a, a), \bar{a}]}} \quad \text{is dual to} \quad c{\uparrow}\,\cfrac{([\bar{a}, \bar{a}], a)}{i{\uparrow}\,\cfrac{([\bar{a}, \bar{a}], (a, a))}{\mathbf{f}}}$$

For symmetric systems the notions of derivation and proof are connected by the following theorem.

Theorem 1.1. (Deduction Theorem).

Let $\mathcal{S}$ be a symmetric system, then there is a derivation $\Delta \left\Vert \mathcal{S} \right. \begin{array}{c} T \\ \\ R \end{array}$ if and only if there is a proof $\begin{array}{c} \Pi \Vert \mathcal{S} \\ [\bar{T}, R] \end{array}$.

Proof. From a given derivation Δ one can construct a proof Π as follows:

$$\begin{array}{c} T \\ \Delta \Vert \mathcal{S} \\ R \end{array} \quad \rightsquigarrow \quad \begin{array}{c} i\downarrow \dfrac{\mathbf{t}}{[\bar{T}, T]} \\ {[\bar{T},\Delta]} \Vert \mathcal{S} \\ {[\bar{T}, R]} \end{array}$$

and for a given proof Π one obtains a derivation Δ by:

$$\begin{array}{c} \Pi \Vert \mathcal{S} \\ {[\bar{T}, R]} \end{array} \quad \rightsquigarrow \quad \begin{array}{c} T \\ (T,\Pi) \Vert \mathcal{S} \\ s\, \dfrac{(T, [\bar{T}, R])}{[R, (T, \bar{T})]} \\ i\uparrow \dfrac{\phantom{[R, (T, \bar{T})]}}{R} \end{array}$$

□

In the calculus of structures the problem of cut elimination from sequent calculus corresponds to the question whether the *up-fragment*, i.e. the set of rules whose names contain ↑-arrows, of a system can be removed while preserving its completeness. The following definitions help to formalise this problem.

Definition 1.11. A rule ρ is *derivable* for a system $\mathcal{S}$ if $\rho \notin \mathcal{S}$ and for every instance $\rho \dfrac{T}{R}$ there is a derivation $\begin{array}{c} T \\ \Delta \Vert \mathcal{S} \\ R \end{array}$.

Definition 1.12. A rule ρ is *admissible* for a system $\mathcal{S}$ if $\rho \notin \mathcal{S}$ and for every proof $\begin{array}{c} \Pi \Vert \mathcal{S}\cup\{\rho\} \\ R \end{array}$ there is a proof $\begin{array}{c} \Pi' \Vert \mathcal{S} \\ R \end{array}$.

Definition 1.13. Two systems $\mathcal{S}$ and $\mathcal{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c} T \\ \Delta \Vert \mathcal{S} \\ R \end{array}$ there is a derivation $\begin{array}{c} T \\ \Delta' \Vert \mathcal{S}' \\ R \end{array}$.

Definition 1.14. Two systems $\mathcal{S}$ and $\mathcal{S}'$ are *(weakly) equivalent* if for every proof $\begin{array}{c} \Pi \Big\| \mathcal{S} \\ R \end{array}$ there is a proof $\begin{array}{c} \Pi' \Big\| \mathcal{S}' \\ R \end{array}$.

It can be shown that the up-fragment of system SKSg is admissible for the system $\mathsf{KSg} = \{i{\downarrow}, s, w{\downarrow}, c{\downarrow}\}$ (see [Brü04]). This implies that system SKSg and the asymmetric and cut-free system KSg are equivalent, and as a consequence of the deduction theorem we get as a result:

$$\text{There is a derivation } \begin{array}{c} T \\ \Delta \Big\| \mathsf{SKSg} \\ R \end{array} \text{ if and only if there is a proof } \begin{array}{c} \Pi \Big\| \mathsf{KSg} \\ [\bar{T}, R] \end{array}.$$

Unfortunately, the two systems are not strongly equivalent, and as a consequence the following does not hold:

$$\text{There is a derivation } \begin{array}{c} T \\ \Delta \Big\| \mathsf{KSg} \\ R \end{array} \text{ if and only if there is a proof } \begin{array}{c} \Pi \Big\| \mathsf{KSg} \\ [\bar{T}, R] \end{array}$$

Therefore the system KSg is used for showing the validity of formulae while the symmetric system SKSg is used when one is interested in derivations.

2 $\mathcal{H}(@)$ in the Calculus of Structures

In [Str07] an inference system in the calculus of structures is presented for the hybrid logic $\mathcal{H}(@)$. This system $\mathsf{BH}{\downarrow}{\uparrow}$ was obtained by simulating the tableau system for $\mathcal{H}(@)$ which is given in [Bla00]. Soundness and completeness of $\mathsf{BH}{\downarrow}{\uparrow}$ follows from the soundness and completeness of the tableau system. System $\mathsf{BH}{\downarrow}{\uparrow}$ will be presented in this section followed by a discussion of its drawbacks.

In order to introduce $\mathsf{BH}{\downarrow}{\uparrow}$, the definitions of structures and contexts have to be extended to the language of $\mathcal{H}(@)$.

Definition 2.1. *Structures* for the logic $\mathcal{H}(@)$ are defined by the grammar

$$R ::= a \mid \mathbf{t} \mid \mathbf{f} \mid \bar{R} \mid (R, R) \mid [R, R] \mid \Box R \mid \Diamond R \mid @_i R$$

where a can be any atom and i is a nominal.

Definition 2.2. *Contexts* for the logic $\mathcal{H}(@)$ are defined by the grammar

$$S ::= \{\} \mid (S, R) \mid [S, R] \mid \Box S \mid \Diamond S \mid @_i S$$

where i is a nominal and R stands for some structure.

Figure 4 shows the extended equivalence relation $=$ on $\mathcal{H}(@)$ structures, which adds De Morgan rules for the modal operators $\Box$ and $\Diamond$ and for the satisfaction operator @. Due to associativity it is also allowed to use conjunction and disjunction in an n-ary form like $(A_1, \ldots, A_n)$ or $[A_1, \ldots, A_n]$ instead of the binary form alone.

The inference rules of system $\mathsf{BH}{\downarrow}{\uparrow}$ are shown in Figure 5. The down-fragment of $\mathsf{BH}{\downarrow}{\uparrow}$, i.e. the set containing the switch rule and all rules marked with $\downarrow$, is denoted by $\mathsf{BH}{\downarrow}$ and analogously the up-fragment of $\mathsf{BH}{\downarrow}{\uparrow}$, i.e. the set containing the switch rule and all rules marked with $\uparrow$, is denoted by $\mathsf{BH}{\uparrow}$.

System $\mathsf{BH}{\downarrow}{\uparrow}$ uses the atomic versions $ai{\downarrow}$ and $ai{\uparrow}$ of the interaction and the cut rule. The non-atomic rules $i{\downarrow}$ and $i{\uparrow}$

$$i{\downarrow}\,\frac{S\{\mathbf{t}\}}{S\{[\bar{A}, A]\}} \qquad\qquad i{\uparrow}\,\frac{S\{(\bar{A}, A)\}}{S\{\mathbf{f}\}}$$

can be shown to be derivable in $\mathsf{BH}{\downarrow}{\uparrow}$ by adapting the inductive proof on the size of the structures A from [Brü06] to cover the modal operators and the satisfaction statements of $\mathcal{H}(@)$ as well. The missing inductive cases for $\mathcal{H}(@)$ are the following:

- $A = \Box B$:

$$k^{\Box}{\downarrow}\,\dfrac{i{\downarrow}\,\dfrac{e^{\Box}{\downarrow}\,\dfrac{S\{\mathbf{t}\}}{S\{\Box\mathbf{t}\}}}{S\{\Box\,[B, \bar{B}]\}}\,(IH)}{S\{[\Box B, \Diamond\bar{B}]\}}$$

Associativity		
	$((A,B),C) = (A,(B,C))$	
	$[[A,B],C] = [A,[B,C]]$	
Commutativity		
	$(A,B) = (B,A)$	
	$[A,B] = [B,A]$	
Units		
	$[\mathbf{f},A] = A$	
	$(\mathbf{t},A) = A$	
Negation		
	$\bar{\mathbf{f}} = \mathbf{t}$	$\bar{\mathbf{t}} = \mathbf{f}$
	$\overline{(A,B)} = [\bar{A},\bar{B}]$	$\overline{[A,B]} = (\bar{A},\bar{B})$
	$\overline{\Box A} = \Diamond\bar{A}$	$\overline{\Diamond A} = \Box\bar{A}$
	$\bar{\bar{A}} = A$	$\overline{@_i A} = @_i\bar{A}$
Context Closure	if $A = B$ then	$S\{A\} = S\{B\}$
		$\bar{A} = \bar{B}$

Figure 4: Equivalence relation $=$ defined on $\mathcal{H}(@)$ structures.

- $A = \Diamond B$:

$$k^{\Box}\downarrow \frac{i\downarrow \dfrac{e^{\Box}\downarrow \dfrac{S\{\mathbf{t}\}}{S\{\Box\mathbf{t}\}}}{S\{\Box[B,\bar{B}]\}}\,(IH)}{S\{[\Diamond B,\Box\bar{B}]\}}$$

- $A = @_i B$:

$$k^{@}\downarrow \frac{i\downarrow \dfrac{e^{@}\downarrow \dfrac{S\{\mathbf{t}\}}{S\{@_i\mathbf{t}\}}}{S\{@_i[B,\bar{B}]\}}\,(IH)}{S\{[@_i B,@_i\bar{B}]\}}$$

$$ai\downarrow \frac{S\{\mathbf{t}\}}{S\{[\bar{a}, a]\}} \qquad s \frac{S\{(A, [B, C])\}}{S\{[(A, B), C]\}} \qquad ai\uparrow \frac{S\{(\bar{a}, a)\}}{S\{\mathbf{f}\}}$$

$$w\downarrow \frac{S\{\mathbf{f}\}}{S\{A\}} \qquad c\downarrow \frac{S\{[A, A]\}}{S\{A\}} \qquad c\uparrow \frac{S\{A\}}{S\{(A, A)\}} \qquad w\uparrow \frac{S\{A\}}{S\{\mathbf{t}\}}$$

$$e^{\Box}\downarrow \frac{S\{\mathbf{t}\}}{S\{\Box\mathbf{t}\}} \qquad k^{\Box}\downarrow \frac{S\{\Box\,[A, B]\}}{S\{[\Box A, \Diamond B]\}} \qquad k^{\Box}\uparrow \frac{S\{(\Box A, \Diamond B)\}}{S\{\Diamond\,(A, B)\}} \qquad e^{\Box}\uparrow \frac{S\{\Diamond\mathbf{f}\}}{S\{\mathbf{f}\}}$$

$$e^{@}\downarrow \frac{S\{\mathbf{t}\}}{S\{@_i\mathbf{t}\}} \qquad k^{@}\downarrow \frac{S\{@_i\,[A, B]\}}{S\{[@_iA, @_iB]\}} \qquad k^{@}\uparrow \frac{S\{(@_iA, @_iB)\}}{S\{@_i\,(A, B)\}} \qquad e^{@}\uparrow \frac{S\{@_i\mathbf{f}\}}{S\{\mathbf{f}\}}$$

$$n\downarrow \frac{S\{@_iA\}}{S\{[\bar{i}, A]\}} \qquad n\uparrow \frac{S\{(i, A)\}}{S\{@_iA\}}$$

$$n^{\Box}\downarrow \frac{S\{@_iA\}}{S\{\Box @_iA\}} \qquad n^{@}\downarrow \frac{S\{@_iA\}}{S\{@_j@_iA\}} \qquad n^{@}\uparrow \frac{S\{@_j@_iA\}}{S\{@_iA\}} \qquad n^{\Box}\uparrow \frac{S\{\Diamond @_iA\}}{S\{@_iA\}}$$

$$r\downarrow \frac{S\{\mathbf{t}\}}{S\{@_ii\}} \qquad \sigma_n\downarrow \frac{S\{@_j\bar{i}\}}{S\{@_i\bar{j}\}} \qquad \sigma_n\uparrow \frac{S\{@_ji\}}{S\{@_ij\}} \qquad r\uparrow \frac{S\{@_i\bar{i}\}}{S\{\mathbf{f}\}}$$

$$v\downarrow^{*} \frac{(C, [[@_i\Box\bar{j}, @_jA], B], D)}{(C, [@_i\Box A, B], D)} \qquad v\uparrow^{*} \frac{[C, (@_i\Diamond A, B), D]}{[C, ((@_i\Diamond j, @_jA), B), D]}$$

$^{*}j$ does not occur in A, B, C or D

$$b\downarrow \frac{S\{@_i\Box\bar{k}\}}{S\{[@_i\Box\bar{j}, @_j\bar{k}]\}} \qquad b\uparrow \frac{S\{(@_i\Diamond j, @_jk)\}}{S\{@_i\Diamond k\}}$$

Figure 5: System BH↓↑ for $\mathcal{H}(@)$.

For the cut rule $i\uparrow$ the proof can be done dually. This result allows us to use the non-atomic rules in derivations as well. Figure 6 shows a proof in system BH↓↑ for one of the example formulae given in Section 1.1.

One drawback of system BH↓↑ is the form of the rules $v\downarrow$ and $v\uparrow$. They cannot be applied within an arbitrary context $S\{\}$, but may only be applied in contexts of a particular form. Furthermore, the nominal j may not occur in the contexts $(C, [\{\}, B], D)$ and $[C, (\{\}, B), D]$ respectively. These conditions stem from the simulation of Blackburn's tableau system. The restricted context reflects the structure of a tableau, and the condition on the nominal j comes from one of the tableau rules which is subject to a corresponding condition. Although we know that the two rules are sound, since the rules of the tableau system were shown to be sound, they do not represent proper implications in the sense that the conclusion is a logical consequence of the premise as it is the case for all of the other inference rules.

$$
\begin{array}{rc}
 & \mathbf{t} \\
i\downarrow & \overline{\left[@_i b, @_i \bar{b}\right]} \\
= & \overline{\left(\mathbf{t}, \left[@_i b, @_i \bar{b}\right]\right)} \\
i\downarrow & \overline{\left(\left[@_i a, @_i \bar{a}\right], \left[@_i b, @_i \bar{b}\right]\right)} \\
s & \overline{\left[@_i \bar{a}, \left(@_i a, \left[@_i b, @_i \bar{b}\right]\right)\right]} \\
s & \overline{\left[@_i \bar{a}, @_i \bar{b}, \left(@_i a, @_i b\right)\right]} \\
k^{@}\uparrow & \overline{\left[@_i \bar{a}, @_i \bar{b}, @_i \left(a, b\right)\right]}
\end{array}
$$

Figure 6: Proof of $(@_i a \wedge @_i b) \supset @_i(a \wedge b)$ in system $\mathsf{BH}\downarrow\uparrow$.

As mentioned in Section 1.2, there is an interest in a system where the whole up-fragment is admissible. However it is not clear yet, whether this is also possible for $\mathcal{H}(@)$. In the simulation of Blackburn's tableau system only inference rules from the set $\mathsf{BH}\uparrow \cup \{k^{@}\downarrow\}$ are used. Tableau systems are refutation-based, i.e. one cannot prove the validity of a formula directly, but one can show whether a formula is unsatisfiable. The validity of a formula can be shown by proving that its negation is unsatisfiable. This search for a refutation in the tableau system is simulated by searching for a refutation in the calculus of structures. The duality between proofs and refutations implies that the system $\mathsf{BH}\downarrow \cup \{k^{@}\uparrow\}$ is complete for $\mathcal{H}(@)$. To obtain a strong cut elimination result, one would have to show additionally that $k^{@}\uparrow$ is admissible for $\mathsf{BH}\downarrow$.

$$
gb\downarrow \frac{S\{@_i \Box A\}}{S\{[@_i \Box \bar{j}, @_j A]\}} \qquad gb\uparrow \frac{S\{(@_i \Diamond j, @_j A)\}}{S\{@_i \Diamond A\}}
$$

$$
gv\downarrow^{*} \frac{S\{[@_i \Box \bar{j}, @_j A]\}}{S\{@_i \Box A\}} \qquad gv\uparrow^{*} \frac{S\{@_i \Diamond A\}}{S\{(@_i \Diamond j, @_j A)\}}
$$

${}^{*}j$ does not occur in A or $S\{\}$

Figure 7: Generalized versions of the v- and b-rules.

A closer look on the v- and b-rules reveals some kind of similarity between $v\downarrow$ and $b\downarrow$ as well as between $v\uparrow$ and $b\uparrow$. In Figure 7 generalized versions of these rules are shown. The rules $gb\downarrow$ and $gb\uparrow$ obviously generalize the rules $b\downarrow$ and $b\uparrow$ by allowing arbitrary structures A instead of the nominal k in $b\uparrow$ or its negation $\bar{k}$ in $b\uparrow$. It can be shown that the rules represent valid implications in $\mathcal{H}(@)$ by proving them in $\mathsf{BH}\downarrow\uparrow$:

$$
\begin{array}{rc}
 & \mathbf{t} \\
i\downarrow & \overline{\left[@_j\bar{A}, @_jA\right]} \\
n^{@}\downarrow & \overline{\left[@_i@_j\bar{A}, @_jA\right]} \\
n^{\Box}\downarrow & \overline{\left[@_i\Box@_j\bar{A}, @_jA\right]} \\
n\downarrow & \overline{\left[@_i\Box\left[\bar{j}, \bar{A}\right], @_jA\right]} \\
k^{\Box}\downarrow & \overline{\left[@_i\left[\Diamond\bar{A}, \Box\bar{j}\right], @_jA\right]} \\
k^{@}\downarrow & \overline{\left[@_i\Diamond\bar{A}, @_i\Box\bar{j}, @_jA\right]}
\end{array}
$$

For $gb\downarrow$ the proof can be done analogously. The rules $gv\downarrow$ and $gv\uparrow$ generalize the rules $v\downarrow$ and $v\uparrow$ by allowing arbitrary contexts $S\{\}$ instead of the restricted form of the contexts in the original rules. Completeness is obviously preserved when the v-rules are replaced by their generalized versions. As the rules $v\downarrow$ and $v\uparrow$, the rules $gv\downarrow$ and $gv\uparrow$ do not incorporate proper implications. However, it can be shown that the soundness of $v\downarrow$ and $v\uparrow$ depends on the condition on the nominal j alone and is independent of the structure of the context. By dropping this restriction of the context one obtains the generalised versions of the rules. The proof is done by a case analysis on the context using Kripke semantics. Consider the rule $v\downarrow$ and let $\mathcal{M}$ be some model and w some state. Furthermore, let R be the premise $(C, [[@_i\Box\bar{j}, @_jA], B], D)$ and T the conclusion $(C, [@_i\Box A, B], D)$ of $v\downarrow$. To show that $v\downarrow$ is a valid implication, one has to prove that $\mathcal{M}, w \Vdash \neg R \vee T$ holds. Note that we can assume that B, C, and D are formulae corresponding to the structures from the contexts of $v\downarrow$ (a formal translation between $\mathcal{H}(@)$ structures and $\mathcal{H}(@)$ formulae is given in a later section). Now we can start the case analysis on B, C, and D:

- If $\mathcal{M}, w \nVdash C$ or $\mathcal{M}, w \nVdash D$, then $\mathcal{M}, w \Vdash \neg R \vee T$.
- If $\mathcal{M}, w \Vdash C$, $\mathcal{M}, w \Vdash D$, and $\mathcal{M}, w \Vdash B$, then $\mathcal{M}, w \Vdash \neg R \vee T$.
- In the remaining case where $\mathcal{M}, w \Vdash C$, $\mathcal{M}, w \Vdash D$, and $\mathcal{M}, w \nVdash B$, $\mathcal{M}, w \Vdash \neg R \vee T$ holds iff $\mathcal{M}, w \Vdash \neg(@_i\Box\neg j \vee @_jA) \vee @_i\Box A$ holds.

The third case is the only one which can make the implication false. For this to happen we must have that $\mathcal{M}, w \nVdash (@_i\Diamond j \wedge @_j\neg A) \vee @_i\Box A$ for some $\mathcal{M}$ and w, but the condition that j does not occur in A or the context $(C, [\{\}, B], D)$, i.e. that j is a new nominal, prevents such a situation. This also shows that $gv\downarrow$ is sound, since it has the same condition that j must be a new nominal. A similar argumentation can be used to justify the soundness of $gv\uparrow$.

Since the generalized versions of the rules can be applied in a more flexible way than the original rules, they will be used in the remainder of this thesis.

3 Translating between BH↓↑ and Sequent Calculus

A standard technique to show the admissibility of the cut-rule in the calculus of structures relies on cut elimination results from sequent calculus. By a translation between the different types of systems it is possible to transfer the cut elimination result from sequent calculus to the calculus of structures as long as certain conditions are fulfilled. In this section, it is explained how this can be done in general. Furthermore, translations between BH↓↑ and two different sequent systems for $\mathcal{H}(@)$ are presented. Unfortunately, none of the two translations allows to transfer the cut elimination result to BH↓↑. But we obtain a way to translate derivations from the calculus of structures into derivations in two different sequent calculae and vice versa. For the first sequent system, this can be done for any derivation and the second system allows the same, as long as there are no instances of gv-rules within the derivation for which the A is instantiated to a nominal.

3.1 Cut Elimination

It can be shown that admissibility of the cut rule implies that the whole up-fragment of a system is admissible.

Lemma 3.1. Every rule $\rho' \dfrac{S\{\bar{Q}\}}{S\{\bar{P}\}}$ is admissible for $\{i{\downarrow}, i{\uparrow}, s, \rho\}$, where $\rho \dfrac{S\{P\}}{S\{Q\}}$ is the dual rule of ρ'.

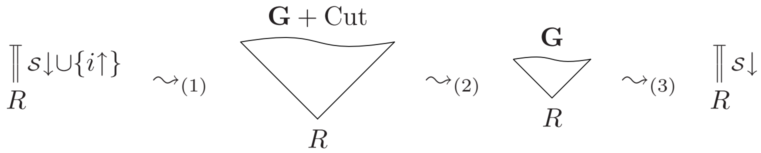
Proof. Replacing every occurence of ρ' in a proof Π by the following derivation yields a proof Π' without ρ'.

$$
\begin{array}{l}
= \dfrac{S\{\bar{Q}\}}{S\{(\bar{Q},\mathbf{t})\}} \\
i{\downarrow} \dfrac{}{S\{(\bar{Q},[P,\bar{P}])\}} \\
s \dfrac{}{S\{[(\bar{Q},P),\bar{P}]\}} \\
\rho \dfrac{}{S\{[(\bar{Q},Q),\bar{P}]\}} \\
i{\uparrow} \dfrac{}{S\{[\mathbf{f},\bar{P}]\}} \\
= \dfrac{}{S\{\bar{P}\}}
\end{array}
$$

□

An immediate consequence of this lemma is that each proof Π for R in a symmetric system $\mathcal{S}{\downarrow}{\uparrow}$ can be turned into a proof Π' for R in $\mathcal{S}{\downarrow}\cup\{i{\uparrow}\}$. The only remaining up-rule is the cut rule $i{\uparrow}$ itself.

One common way to show the admissibility of the cut rule in the calculus of structures is to use cut elimination results from the sequent calculus. This is done by showing that each proof in the calculus of structures can be translated into a proof in sequent calculus which may contain cuts, then cut elimination is applied to the proof in the sequent calculus and finally, the proof is translated back into a cut-free proof in the calculus of structures. The idea is illustrated by the following figure:



where **G** is some cut-free sequent system for the same logic as $\mathcal{S}{\downarrow}{\uparrow}$. In detail, this kind of cut elimination proof consists of the following steps:

1. A proof in the system $\mathcal{S}{\downarrow}\cup\{i{\uparrow}\}$ is translated into a proof in the sequent calculus $\mathbf{G} + \text{Cut}$ which may contain cuts.
2. The cut elimination procedure for $\mathbf{G} + \text{Cut}$ is applied to obtain a cut-free proof in **G**.
3. The cut-free proof in **G** is translated back into system $\mathcal{S}{\downarrow}$. This step only works, if no up-rules of $\mathcal{S}{\downarrow}{\uparrow}$ are introduced during the translation.

3.2 The Sequent Calculus $\mathbf{G}_{\mathcal{H}(@)}$ for Hybrid Logic

According to [TS96], a two-sided sequent calculus can be transformed into an equivalent one-sided calculus, a so-called Gentzen-Schütte system. Here, a cut-free system from [Bra08] (Chapter 2) is transformed into a one-sided system which is more suitable for translation between sequent calculus and calculus of structures. The system was obtained by negating for each rule the sequents on the left side of $\vdash$ and moving them to the right side. Instead of the implication rules from the original system, a rule for disjunction is used which better resembles the rules from the calculus of structures. Furthermore, the contexts were adapted and rules for weakening and contraction were added to allow multiplicative instead of additive context treatment. Figure 8 shows the resulting one-sided sequent calculus $\mathbf{G}_{\mathcal{H}(@)}$ for $\mathcal{H}(@)$.

The system $\mathbf{G}_{\mathcal{H}(@)} + \text{Cut}$ is obtained by adding the following cut rule to $\mathbf{G}_{\mathcal{H}(@)}$.

$$\text{(cut)}\ \frac{\vdash \Phi, A \qquad \vdash \Psi, \neg A}{\vdash \Phi, \Psi}$$

Note that the system only works when all formulae occuring in a proof or a derivation are satisfaction statements, i.e. all formulae are prefixed with $@_i$ for

$$(\text{axiom})\ \frac{}{\vdash A, \neg A} \qquad (\top)\ \frac{}{\vdash @_i \top}$$

$$(\wedge R)\ \frac{\vdash \Phi, @_i A \qquad \vdash \Psi, @_i B}{\vdash \Phi, \Psi, @_i(A \wedge B)} \qquad (\vee R)\ \frac{\vdash \Phi, @_i A, @_i B}{\vdash \Phi, @_i(A \vee B)}$$

$$(CR)\ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad (WR)\ \frac{\vdash \Phi}{\vdash \Phi, A}$$

$$(\Box R)^{*}\ \frac{\vdash \Phi, @_i \Box \neg j, @_j A}{\vdash \Phi, @_i \Box A} \qquad (\Diamond R)\ \frac{\vdash \Phi, @_i \Diamond j \qquad \vdash \Psi, @_j A}{\vdash \Phi, \Psi, @_i \Diamond A}$$

$$(@R)\ \frac{\vdash \Phi, @_i A}{\vdash \Phi, @_j @_i A} \qquad (\text{ref})\ \frac{\vdash \Phi, @_i \neg i}{\vdash \Phi}$$

$$(\text{nom1})\ \frac{\vdash \Phi, @_i j \qquad \vdash \Psi, @_i A}{\vdash \Phi, \Psi, @_j A}$$

$$(\text{nom2})\ \frac{\vdash \Gamma, @_i j \qquad \vdash \Phi, @_i \Diamond k \qquad \vdash \Psi, @_j \Box \neg k}{\vdash \Gamma, \Phi, \Psi}$$

* j does not appear free in the conclusion

Figure 8: The sequent calculus $\mathbf{G}_{\mathcal{H}(@)}$ for $\mathcal{H}(@)$.

some nominal i. In the original system by Braüner there is a condition on the ($nom1$)-rule which requires that A must be an atom (propositional variable or nominal). This condition allows to obtain a normalisation result which otherwise would not be possible (cf. [Bra08], p.39), but it is not needed for the soundness of the rule.

3.2.1 Translation from $\mathbf{G}_{\mathcal{H}(@)}$ to $\mathsf{BH}{\downarrow}{\uparrow}$

The following definition of $\underline{\ \cdot\ }_S$ allows to recursively translate formulae and sequents from sequent calculus into structures in the calculus of structures. The assumption that all negation signs are pushed down to the atoms also applies to the formulae in the sequent calculus.

Definition 3.1. Mapping $\underline{\ \cdot\ }_S$ from formulae and sequents in $\mathbf{G}_{\mathcal{H}(@)}$ to structures in $\mathsf{BH}{\downarrow}{\uparrow}$:

$$
\begin{array}{lcl}
\underline{a}_S & = & a \\
\underline{\top}_S & = & \mathbf{t} \\
\underline{\bot}_S & = & \mathbf{f} \\
\underline{(A \vee B)}_S & = & [\underline{A}_S, \underline{B}_S] \\
\underline{(A \wedge B)}_S & = & (\underline{A}_S, \underline{B}_S) \\
\underline{\Box A}_S & = & \Box \underline{A}_S \\
\underline{\Diamond A}_S & = & \Diamond \underline{A}_S \\
\underline{@_i A}_S & = & @_i \underline{A}_S \\
\underline{\varnothing}_S & = & \mathbf{f} \\
\underline{A_1, \ldots, A_h}_S & = & \left[\underline{A_1}_S, \ldots, \underline{A_h}_S\right] \text{ where } h > 0
\end{array}
$$

Now the translation from sequent calculus to the calculus of structures can be formalised.

Theorem 3.1. For every derivation $\dfrac{\Sigma_1 \;\ldots\; \Sigma_k}{\Sigma}\,\Delta$ in $\mathbf{G}_{\mathcal{H}(@)}$ + Cut there is a derivation $\begin{array}{c}(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S) \\ \Delta' \Big\Vert \\ \underline{\Sigma}_S\end{array}$ in $\mathsf{BH}{\downarrow}{\uparrow}$.

Proof. Structural induction on the derivation Δ.

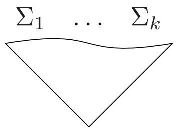
Base cases:

- $\Delta = \Sigma$. Take $\underline{\Sigma}_S$.
- $\Delta = (\top)\,\dfrac{}{\vdash @_i \top}$. Take $e^{@}{\downarrow}\,\dfrac{\mathbf{t}}{@_i \mathbf{t}}$.
- $\Delta = (\text{axiom})\,\dfrac{}{\vdash A, \neg A}$. Take $i{\downarrow}\,\dfrac{\mathbf{t}}{[\underline{A}_S, \bar{\underline{A}}_S]}$.

Inductive cases:

- $\Delta = (\wedge R)\,\dfrac{\dfrac{\Sigma_1 \;\ldots\; \Sigma_k}{\vdash \Phi, @_i A} \qquad \dfrac{\Sigma'_1 \;\ldots\; \Sigma'_l}{\vdash \Psi, @_i B}}{\vdash \Phi, \Psi, @_i(A \wedge B)}$. By induction hypothesis we have two derivations $\begin{array}{c}(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S) \\ \Delta_1 \Big\Vert \mathsf{BH}{\downarrow}{\uparrow} \\ [\underline{\Phi}_S, @_i \underline{A}_S]\end{array}$ and $\begin{array}{c}(\underline{\Sigma'_1}_S, \ldots, \underline{\Sigma'_l}_S) \\ \Delta_2 \Big\Vert \mathsf{BH}{\downarrow}{\uparrow} \\ [\underline{\Psi}_S, @_i \underline{B}_S]\end{array}$ which are plugged into the contexts $\Delta_1' = (\Delta_1, \underline{\Sigma_1'}_S, \ldots, \underline{\Sigma_l'}_S)$ and $\Delta_2' = ([\underline{\Phi}_S, @_i\underline{A}_S], \Delta_2)$ to obtain the derivation

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S, \underline{\Sigma_1'}_S, \ldots, \underline{\Sigma_l'}_S) \\ \Delta_1' \Big\| \mathsf{BH}{\downarrow}{\uparrow} \\ ([\underline{\Phi}_S, @_i\underline{A}_S], \underline{\Sigma_1'}_S, \ldots, \underline{\Sigma_l'}_S) \\ \Delta_2' \Big\| \mathsf{BH}{\downarrow}{\uparrow} \\ ([\underline{\Phi}_S, @_i\underline{A}_S], [\underline{\Psi}_S, @_i\underline{B}_S])\end{array}}{[\underline{\Psi}_S, ([\underline{\Phi}_S, @_i\underline{A}_S], @_i\underline{B}_S)]}\, s}{[\underline{\Phi}_S, \underline{\Psi}_S, (@_i\underline{A}_S, @_i\underline{B}_S)]}\, s}{[\underline{\Phi}_S, \underline{\Psi}_S, @_i(\underline{A}_S, \underline{B}_S)]}\, k^{@}{\uparrow}$$

- $\Delta =$  . By induction hypothesis we have a derivation

$$(\vee R)\ \dfrac{\vdash \Phi, @_iA, @_iB}{\vdash \Phi, @_i(A \vee B)}$$

$$\begin{array}{c}(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S) \\ \Delta \Big\| \mathsf{BH}{\downarrow}{\uparrow} \\ [\underline{\Phi}_S, @_i\underline{A}_S, @_i\underline{B}_S]\end{array}$$ which leads to the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S) \\ \Delta \Big\| \mathsf{BH}{\downarrow}{\uparrow} \\ [\underline{\Phi}_S, @_i\underline{A}_S, @_i\underline{B}_S]\end{array}}{[\underline{\Phi}_S, @_i[\underline{A}_S, \mathbf{f}], @_i[\underline{B}_S, \mathbf{f}]]}\, =}{[\underline{\Phi}_S, @_i[\underline{A}_S, \underline{B}_S], @_i[\mathbf{f}, \underline{B}_S]]}\, w{\downarrow}}{[\underline{\Phi}_S, @_i[\underline{A}_S, \underline{B}_S], @_i[\underline{A}_S, \underline{B}_S]]}\, w{\downarrow}}{[\underline{\Phi}_S, @_i[\underline{A}_S, \underline{B}_S]]}\, c{\downarrow}$$

The argumentation for branching rules is similar to the one for the $(\wedge R)$-rule and for non-branching rules the argumentation follows the one for the $(\vee R)$-rule. Therefore, only the relevant part of the derivation is shown for the remaining rules.

- (CR) leads to

$$c{\downarrow}\ \dfrac{[\underline{\Phi}_S, \underline{A}_S, \underline{A}_S]}{[\underline{\Phi}_S, \underline{A}_S]}$$

- (WR) leads to

$$\dfrac{\dfrac{\underline{\Phi}_S}{[\underline{\Phi}_S, \mathbf{f}]}\, =}{[\underline{\Phi}_S, \underline{A}_S]}\, w{\downarrow}$$

- $(\Box R)$ leads to

$$gv\downarrow\,\dfrac{[\underline{\Phi}_S, @_i\Box\neg j, @_j\underline{A}_S]}{[\underline{\Phi}_S, @_i\Box\underline{A}_S]}$$

The condition of the $(\Box R)$-rule ensures that the condition of the $gv\downarrow$-rule is fulfilled.

- $(\Diamond R)$ leads to

$$gb\uparrow\,\dfrac{s\,\dfrac{s\,\dfrac{([\underline{\Phi}_S, @_i\Diamond j], [\underline{\Psi}_S, @_j\underline{A}_S])}{[\underline{\Phi}_S, (@_i\Diamond j, [\underline{\Psi}_S, @_j\underline{A}_S])]}}{[\underline{\Phi}_S, \underline{\Psi}_S, (@_i\Diamond j, @_j\underline{A}_S)]}}{[\underline{\Phi}_S, \underline{\Psi}_S, @_i\Diamond\underline{A}_S]}$$

- $(@R)$ leads to

$$n^{@}\downarrow\,\dfrac{[\underline{\Phi}_S, @_i\underline{A}_S]}{[\underline{\Phi}_S, @_j@_i\underline{A}_S]}$$

- (ref) leads to

$$=\,\dfrac{r\uparrow\,\dfrac{[\underline{\Phi}_S, @_i\bar{i}]}{[\underline{\Phi}_S, \mathbf{f}]}}{\underline{\Phi}_S}$$

- $(nom1)$ leads to

$$n^{@}\uparrow\,\dfrac{n\uparrow\,\dfrac{k^{@}\uparrow\,\dfrac{s\,\dfrac{s\,\dfrac{([\underline{\Phi}_S, @_ij], [\underline{\Psi}_S, @_i\underline{A}_S])}{[\underline{\Phi}_S, (@_ij, [\underline{\Psi}_S, @_i\underline{A}_S])]}}{[\underline{\Phi}_S, \underline{\Psi}_S, (@_ij, @_i\underline{A}_S)]}}{[\underline{\Phi}_S, \underline{\Psi}_S, @_i(j, \underline{A}_S)]}}{[\underline{\Phi}_S, \underline{\Psi}_S, @_i@_j\underline{A}_S]}}{[\underline{\Phi}_S, \underline{\Psi}_S, @_j\underline{A}_S]}$$

- $(nom2)$ leads to

$$=\,\dfrac{i\uparrow\,\dfrac{n^{@}\uparrow\,\dfrac{n\uparrow\,\dfrac{k^{@}\uparrow\,\dfrac{s\,\dfrac{s\,\dfrac{s\,\dfrac{s\,\dfrac{([\underline{\Gamma}_S, @_ij], [\underline{\Phi}_S, @_i\Diamond k], [\underline{\Psi}_S, @_j\Box\neg k])}{[\underline{\Psi}_S, ([\underline{\Gamma}_S, @_ij], [\underline{\Phi}_S, @_i\Diamond k], @_j\Box\neg k)]}}{[\underline{\Psi}_S, ([(([\underline{\Gamma}_S, @_ij], @_i\Diamond k), \underline{\Phi}_S], @_j\Box\neg k)]}}{[\underline{\Psi}_S, ([(@_ij, @_i\Diamond k), [\underline{\Gamma}_S, \underline{\Phi}_S]], @_j\Box\neg k)]}}{[\underline{\Gamma}_S, \underline{\Phi}_S, \underline{\Psi}_S, (@_ij, @_i\Diamond k, @_j\Box\neg k)]}}{[\underline{\Gamma}_S, \underline{\Phi}_S, \underline{\Psi}_S, (@_i(j, \Diamond k), @_j\Box\neg k)]}}{[\underline{\Gamma}_S, \underline{\Phi}_S, \underline{\Psi}_S, (@_i@_j\Diamond k, @_j\Box\neg k)]}}{[\underline{\Gamma}_S, \underline{\Phi}_S, \underline{\Psi}_S, (@_j\Diamond k, @_j\Box\neg k)]}}{[\underline{\Gamma}_S, \underline{\Phi}_S, \underline{\Psi}_S, \mathbf{f}]}}{[\underline{\Gamma}_S, \underline{\Phi}_S, \underline{\Psi}_S]}$$

- (cut) leads to

$$
s\,\frac{s\,\frac{i\!\uparrow\frac{=\frac{[\underline{\Phi}_S,\underline{\Psi}_S,\mathbf{f}]}{[\underline{\Phi}_S,\underline{\Psi}_S]}}{[\underline{\Phi}_S,\underline{\Psi}_S,(\underline{A}_S,\neg\underline{A}_S)]}}{[\underline{\Phi}_S,(\underline{A}_S,[\underline{\Psi}_S,\neg\underline{A}_S])]}}{([\underline{\Phi}_S,\underline{A}_S],[\underline{\Psi}_S,\neg\underline{A}_S])}
$$

$$
\begin{array}{rc}
s & \dfrac{([\underline{\Phi}_S,\underline{A}_S],[\underline{\Psi}_S,\neg\underline{A}_S])}{[\underline{\Phi}_S,(\underline{A}_S,[\underline{\Psi}_S,\neg\underline{A}_S])]} \\
s & \dfrac{}{[\underline{\Phi}_S,\underline{\Psi}_S,(\underline{A}_S,\neg\underline{A}_S)]} \\
i\!\uparrow & \dfrac{}{[\underline{\Phi}_S,\underline{\Psi}_S,\mathbf{f}]} \\
= & \dfrac{}{[\underline{\Phi}_S,\underline{\Psi}_S]}
\end{array}
$$

□

The stronger result that for every derivation in $\mathbf{G}_{\mathcal{H}(@)}$ there is a corresponding derivation in $\mathsf{BH}\!\downarrow$ (similar to the result given in [Brü04] for classical logic) is not achieved for $\mathcal{H}(@)$, since not only the translation of the (cut)-rule introduces the $i\!\uparrow$-rule but also the $(nom2)$-rule. Furthermore, the translations of some of the other rules, namely the rules $(\wedge R)$, $(\Diamond R)$, (ref), $(nom1)$, and $(nom2)$, introduce more rules from the up-fragment of $\mathsf{BH}\!\downarrow\!\uparrow$. For that reason, the translation between $\mathbf{G}_{\mathcal{H}(@)}$ and $\mathsf{BH}\!\downarrow\!\uparrow$ does not prove cut elimination as desired. However, we obtain a different result. Starting with system $\mathbf{G}_{\mathcal{H}(@)}$, which is complete for $\mathcal{H}(@)$, we obtain proofs in $\mathsf{BH}\!\downarrow\!\uparrow$ which only use the rules $i\!\downarrow$, s, $c\!\downarrow$, $w\!\downarrow$, $n^@\!\downarrow$, $gv\!\downarrow$, $e^@\!\downarrow$, $i\!\uparrow$, $r\!\uparrow$, $k^@\!\uparrow$, $n\!\uparrow$, $n^@\!\uparrow$, and $gb\!\uparrow$. Together with Lemma 3.1 this yields:

Lemma 3.2. The system $\{i\!\uparrow, i\!\downarrow, s, c\!\downarrow, w\!\downarrow, n^@\!\downarrow, e^@\!\downarrow, r\!\downarrow, k^@\!\downarrow, n\!\downarrow, gb\!\downarrow, gv\!\downarrow\}$ is complete for $\mathcal{H}(@)$, and the rules $e^\Box\!\downarrow$, $k^\Box\!\downarrow$, $n^\Box\!\downarrow$, and $\sigma_n\!\downarrow$ as well as their dual rules are admissible for this system.

3.2.2 Translation from $\mathsf{BH}\!\downarrow\!\uparrow$ to $\mathbf{G}_{\mathcal{H}(@)}$

Even though we are not able to show cut elimination by translation from system $\mathbf{G}_{\mathcal{H}(@)}$ to $\mathsf{BH}\!\downarrow\!\uparrow$, it is interesting whether it is possible to translate derivations in the other direction as well. The translation from the calculus of structures to the sequent calculus also follows the approach of [Brü04] for classical logic. For the following definition, recall that negations in $\mathsf{BH}\!\downarrow\!\uparrow$ only occur on atoms and that by associativity disjunction as well as conjunction can be assumed to be in binary form.

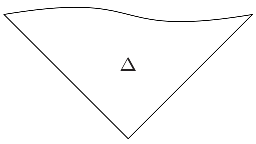
Definition 3.2. Mapping $\underline{\ \cdot\ }_G$ from structures in $\mathsf{BH}\!\downarrow\!\uparrow$ to formulae in $\mathbf{G}_{\mathcal{H}(@)}$:

$$
\begin{array}{lcl}
\underline{a}_G & = & a \\
\underline{\mathbf{t}}_G & = & \top \\
\underline{\mathbf{f}}_G & = & \bot \\
\underline{[A,B]}_G & = & (\underline{A}_G \vee \underline{B}_G) \\
\underline{(A,B)}_G & = & (\underline{A}_G \wedge \underline{B}_G) \\
\underline{\Box A}_G & = & \Box \underline{A}_G \\
\underline{\Diamond A}_G & = & \Diamond \underline{A}_G \\
\underline{@_i A}_G & = & @_i \underline{A}_G
\end{array}
$$

For better readability the mapping $\underline{\ \cdot\ }_G$ is not always shown explicitly. In addition to the translation from structures to formulae in the sequent calculus, we need a way to imitate deep inference in sequent calculus. Therefore, the following lemma is needed.

Lemma 3.3. For every two formulae A, B and every context $C\{\}$ there exists a derivation

$$\vdash @_j A, @_j \neg B$$



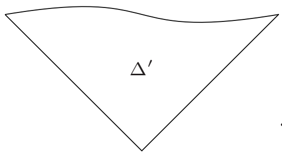
$$\vdash @_i C\{A\}, @_i \neg C\{B\}$$

in $\mathbf{G}_{\mathcal{H}(@)}$.

Proof. By structural induction on the context $C\{\}$. The base case for $C\{\} = \{\}$ is trivial. Inductive cases:

- For $C\{\} = \Box C_1\{\}$ the derivation is

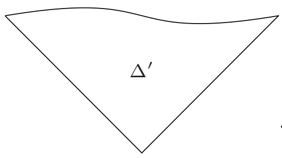
$\Delta =$



$$(\Box R)\dfrac{(\Diamond R)\dfrac{(axiom)\dfrac{}{\vdash @_i \Box \neg j, @_i \Diamond j} \quad \vdash @_j C_1\{A\}, @_j \neg C_1\{B\}}{\vdash @_i \Box \neg j, @_j C_1\{A\}, @_i \Diamond \neg C_1\{B\}}}{\vdash @_i \Box C_1\{A\}, @_i \Diamond \neg C_1\{B\}}.$$

- For $C\{\} = \Diamond C_1\{\}$ the derivation is

$\Delta =$



$$(\Box R)\dfrac{(\Diamond R)\dfrac{(axiom)\dfrac{}{\vdash @_i \Diamond j, @_i \Box \neg j} \quad \vdash @_j C_1\{A\}, @_j \neg C_1\{B\}}{\vdash @_i \Diamond C_1\{A\}, @_i \Box \neg j, @_j \neg C_1\{B\}}}{\vdash @_i \Diamond C_1\{A\}, @_i \Box \neg C_1\{B\}}.$$

- For $C\{\} = @_j C_1\{\}$ the derivation is

$$\Delta = \dfrac{(@R)\;\dfrac{\begin{array}{c}\Delta' \\ \vdots \\ \vdash @_j C_1\{A\}, @_j \neg C_1\{B\}\end{array}}{\vdash @_j C_1\{A\}, @_i @_j \neg C_1\{B\}}}{(@R)\;\vdash @_i @_j C_1\{A\}, @_i @_j \neg C_1\{B\}} .$$

- For $C\{\} = (C_1 \wedge C_2\{\})$ the derivation is

$$\Delta = \dfrac{(\vee R)\;\dfrac{(\wedge R)\;\dfrac{(axiom)\;\dfrac{}{\vdash @_i C_1, @_i \neg C_1} \quad \begin{array}{c}\Delta' \\ \vdots \\ \vdash @_i C_2\{A\}, @_i \neg C_2\{B\}\end{array}}{\vdash @_i (C_1 \wedge C_2\{A\}), @_i \neg C_1, @_i \neg C_2\{B\})}}{\vdash @_i (C_1 \wedge C_2\{A\}), @_i (\neg C_1 \vee \neg C_2\{B\})}}{} .$$

- For $C\{\} = (C_1 \vee C_2\{\})$ the derivation is

$$\Delta = \dfrac{(\vee R)\;\dfrac{(\wedge R)\;\dfrac{(axiom)\;\dfrac{}{\vdash @_i C_1, @_i \neg C_1} \quad \begin{array}{c}\Delta' \\ \vdots \\ \vdash @_i C_2\{A\}, @_i \neg C_2\{B\}\end{array}}{\vdash @_i C_1, @_i C_2\{A\}, @_i (\neg C_1 \wedge \neg C_2\{B\})}}{\vdash @_i (C_1 \vee C_2\{A\}), @_i (\neg C_1 \wedge \neg C_2\{B\})}}{} .$$

The derivations marked with Δ' exist by induction hypothesis. For applications of the $(\Box R)$-rule, the nominal j can always be chosen in such a way that it does not occur freely in the conclusion by taking a new nominal. □

Now we can translate derivations in $\mathsf{BH}{\downarrow}{\uparrow}$ into derivations in $\mathbf{G}_{\mathcal{H}(@)}$.

Theorem 3.2. For every derivation $\Delta \begin{array}{c} Q \\ \Vert \\ P \end{array}$ in $\mathsf{BH}{\downarrow}{\uparrow}$ there is a derivation $\begin{array}{c} \vdash @_j \underline{Q}_G \\ \Delta' \\ \vdash @_i \underline{P}_G \end{array}$ in $\mathbf{G}_{\mathcal{H}(@)}$.

Proof. The derivation Δ' in $\mathbf{G}_{\mathcal{H}(@)}$ is constructed by induction on the length of the derivation in $\mathsf{BH}{\downarrow}{\uparrow}$.

Base Case
If Δ is the trivial derivation consisting of a single structure P, i.e. P and Q coincide, then the corresponding derivation is $\vdash @_i \underline{P}_G$ for some nominal i.

Inductive Case
For the inductive case the topmost rule instance in Δ is singled out:

$$\begin{array}{c} Q \\ \Delta \,\Big\| \,\mathsf{BH}{\downarrow}{\uparrow} \\ P \end{array} = \begin{array}{c} \rho\,\dfrac{S\{T\}}{S\{R\}} \\ \Delta_0 \,\Big\| \,\mathsf{BH}{\downarrow}{\uparrow} \\ P \end{array}$$

Now the corresponding derivation in $\mathbf{G}_{\mathcal{H}(@)}$ is constructed as follows:

$$\begin{array}{c} \Pi \\ \vdots \\ \vdash @_k R, @_k \neg T \\ \Delta_1 \\ \vdots \\ (cut)\,\dfrac{\vdash @_j S\{R\}, @_j \neg S\{T\} \quad \vdash @_j S\{T\}}{\vdash @_j S\{R\}}\,, \\ \Delta_2 \\ \vdots \\ \vdash @_i P \end{array}$$

where Δ_1 exists by Lemma 3.3 and Δ_2 by induction hypothesis. The proof Π has to be shown for each rule in $\mathsf{BH}{\downarrow}{\uparrow}$ individually:

- $i{\downarrow}$:

$$(WR)\,\dfrac{(\vee R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i A, @_i \neg A}}{\vdash @_i (A \vee \neg A)}}{\vdash @_i (A \vee \neg A), @_i \bot}$$

- s:

$$(\vee R)\,\dfrac{(\vee R)\,\dfrac{(\wedge R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A}\qquad (\wedge R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iB, @_i\neg B}\qquad (\mathit{axiom})\,\dfrac{}{\vdash @_iC, @_i\neg C}}{\vdash @_iB, @_iC, @_i(\neg B\wedge\neg C)}}{\vdash @_i(A\wedge B), @_iC, @_i\neg A, @_i(\neg B\wedge\neg C)}}{\vdash @_i(A\wedge B), @_iC, @_i(\neg A\vee(\neg B\wedge\neg C))}}{\vdash @_i((A\wedge B)\vee C), @_i(\neg A\vee(\neg B\wedge\neg C))}$$

- $w{\downarrow}$:

$$(WR)\,\dfrac{(\top)\,\dfrac{}{\vdash @_i\top}}{\vdash @_iA, @_i\top}$$

- $c{\downarrow}$:

$$(CR)\,\dfrac{(\wedge R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A}\qquad (\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A}}{\vdash @_iA, @_iA, @_i(\neg A\wedge\neg A)}}{\vdash @_iA, @_i(\neg A\wedge\neg A)}$$

- $e^{\Box}{\downarrow}$:

$$(WR)\,\dfrac{(\Box R)\,\dfrac{(WR)\,\dfrac{(\top)\,\dfrac{}{\vdash @_j\top}}{\vdash @_i\Box\neg j, @_j\top}}{\vdash @_i\Box\top}}{\vdash @_i\Box\top, @_i\bot}$$

- $k^{\Box}{\downarrow}$:

$$(\vee R)\,\dfrac{(\Box R)\,\dfrac{(CR)\,\dfrac{(\Diamond R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_i\Box\neg j, @_i\Diamond j}\qquad \Pi'}{\vdash @_i\Box\neg j, @_i\Box\neg j, @_jA, @_i\Diamond B, @_i\Diamond(\neg A\wedge\neg B)}}{\vdash @_i\Box\neg j, @_jA, @_i\Diamond B, @_i\Diamond(\neg A\wedge\neg B)}}{\vdash @_i\Box A, @_i\Diamond B, @_i\Diamond(\neg A\wedge\neg B)}}{\vdash @_i(\Box A\vee\Diamond B), @_i\Diamond(\neg A\wedge\neg B)}$$

where Π' is the proof

$$(\Diamond R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_i\Box\neg j, @_i\Diamond j}\qquad (\wedge R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_jA, @_j\neg A}\qquad (\mathit{axiom})\,\dfrac{}{\vdash @_jB, @_j\neg B}}{\vdash @_jA, @_jB, @_j(\neg A\wedge\neg B)}}{\vdash @_jA, @_jB, @_i\Box\neg j, @_i\Diamond(\neg A\wedge\neg B)}$$

- $e^{@}{\downarrow}$:

$$(WR)\,\dfrac{(@R)\,\dfrac{(\top)\,\dfrac{}{\vdash @_i\top}}{\vdash @_j@_i\top}}{\vdash @_j@_i\top, @_j\bot}$$

- $k^{@}{\downarrow}$:

$$(\vee R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(\wedge R)\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A}\qquad (\mathit{axiom})\,\dfrac{}{\vdash @_iB, @_i\neg B}}{\vdash @_iA, @_iB, @_i(\neg A\wedge\neg B)}}{\vdash @_iA, @_iB, @_j@_i(\neg A\wedge\neg B)}}{\vdash @_iA, @_j@_iB, @_j@_i(\neg A\wedge\neg B)}}{\vdash @_j@_iA, @_j@_iB, @_j@_i(\neg A\wedge\neg B)}}{\vdash @_j(@_iA\vee @_iB), @_j@_i(\neg A\wedge\neg B)}$$

- $n{\downarrow}$:

$$(\vee R)\,\dfrac{(@R)\,\dfrac{(nom1)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_j\neg i, @_j i} \qquad (axiom)\,\dfrac{}{\vdash @_j A, @_j\neg A}}{\vdash @_j\neg i, @_j A, @_i\neg A}}{\vdash @_j\neg i, @_j A, @_j @_i \neg A}}{\vdash @_j(\neg i \vee A), @_j @_i\neg A}$$

- $n^{\Box}{\downarrow}$:

$$(\Box R)\,\dfrac{(WR)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i A, @_i\neg A}}{\vdash @_i A, @_j @_i\neg A}}{\vdash @_k @_i A, @_j @_i \neg A}}{\vdash @_j \Box\neg k, @_k @_i A, @_j @_i\neg A}}{\vdash @_j \Box @_i A, @_j @_i \neg A}$$

The nominal k can be chosen in such a way that it does not occur in A.

- $n^{@}{\downarrow}$:

$$(@R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i A, @_i\neg A}}{\vdash @_i A, @_k @_i\neg A}}{\vdash @_j @_i A, @_k @_i\neg A}}{\vdash @_k @_j @_i A, @_k @_i \neg A}$$

- $r{\downarrow}$:

$$(WR)\,\dfrac{(@R)\,\dfrac{(ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i i, @_i\neg i}}{\vdash @_i i}}{\vdash @_j @_i i}}{\vdash @_j @_i i, @_j \bot}$$

- $\sigma_n{\downarrow}$:

$$(@R)\,\dfrac{(nom1)\,\dfrac{(@R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i\neg j, @_i j}}{\vdash @_j @_i \neg j, @_i j} \qquad (ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i i, @_i\neg i}}{\vdash @_i i}}{\vdash @_j @_i\neg j, @_j i}}{\vdash @_j @_i \neg j, @_j @_j i}$$

- $gb{\downarrow}$:

$$(@R)\,\dfrac{(\vee R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(\Diamond R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i \Box\neg j, @_i \Diamond j} \qquad (axiom)\,\dfrac{}{\vdash @_j A, @_j\neg A}}{\vdash @_i\Box\neg j, @_j A, @_i \Diamond\neg A}}{\vdash @_i \Box\neg j, @_k @_j A, @_i\Diamond\neg A}}{\vdash @_k @_i \Box\neg j, @_k @_j A, @_i \Diamond\neg A}}{\vdash @_k(@_i\Box\neg j \vee @_j A), @_i\Diamond\neg A}}{\vdash @_k(@_i \Box\neg j \vee @_j A), @_k @_i \Diamond\neg A}$$

- $gv{\downarrow}$:

$$
(@R)\,\dfrac{(CR)\,\dfrac{(\wedge R)\,\dfrac{(\Box R)\,\dfrac{(WR)\,\dfrac{(@R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i\Box\neg j, @_i\Diamond j}}{\vdash @_i\Box\neg j, @_j@_i\Diamond j}}{\vdash @_i\Box\neg j, @_jA, @_j@_i\Diamond j}}{\vdash @_i\Box A, @_j@_i\Diamond j} \qquad (\Box R)\,\dfrac{(WR)\,\dfrac{(@R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_jA, @_j\neg A}}{\vdash @_jA, @_j@_j\neg A}}{\vdash @_i\Box\neg j, @_jA, @_j@_j\neg A}}{\vdash @_i\Box A, @_j@_j\neg A}}{\vdash @_i\Box A, @_i\Box A, @_j(@_i\Diamond j \wedge @_j\neg A)}}{\vdash @_i\Box A, @_j(@_i\Diamond j \wedge @_j\neg A)}}{\vdash @_j@_i\Box A, @_j(@_i\Diamond j \wedge @_j\neg A)}
$$

We can assume that the nominal j does not occur freely in A, since the $gv{\downarrow}$-rule may only be applied in $\mathsf{BH}{\downarrow}{\uparrow}$ if this is the case. By binding the only remaining free occurrence of j with $@_j$, we can ensure that the condition of the $(\Box R)$-rule is fulfilled for both of its applications.

The proofs for the up-rules can be found analogously. □

During the translation from $\mathsf{BH}{\downarrow}{\uparrow}$ to $\mathbf{G}_{\mathcal{H}(@)}$ the $(nom2)$-rule is not used. Completeness of $\mathsf{BH}{\downarrow}{\uparrow}$ implies that the $(nom2)$-rule is not needed for the completeness of $\mathbf{G}_{\mathcal{H}(@)}$. The reason for this is that in Braüner's original system the rule is needed because of his stronger condition on the $(nom1)$-rule. That means for the system as it is presented here, that the $(nom2)$-rule could also be dropped without losing completeness.

After it was clear that the translation via $\mathbf{G}_{\mathcal{H}(@)}$ does not lead to the desired goal of showing cut elimination for $\mathsf{BH}{\downarrow}{\uparrow}$, a second sequent system for $\mathcal{H}(@)$ was examined. The result of this attempt is shown in the next section.

3.3 The Sequent Calculus $\mathbf{G}'_{\mathcal{H}(@)}$ for Hybrid Logic

Figure 9 shows a one-sided sequent calculus $\mathbf{G}'_{\mathcal{H}(@)}$ based on another sequent system from [Bra08] (Chapter 3) and was constructed in a similar way as $\mathbf{G}_{\mathcal{H}(@)}$ in the previous section.

The system $\mathbf{G}'_{\mathcal{H}(@)}$ is much closer to Blackburn's sequent system and hence, also closer to the corresponding tableau system [Bla00] from which $\mathsf{BH}{\downarrow}{\uparrow}$ was constructed. The reason why $\mathbf{G}'_{\mathcal{H}(@)}$ was favoured over Blackburn's sequent system is that Braüner gives a much more detailed discussion of the properties of his system. Note that system $\mathbf{G}'_{\mathcal{H}(@)}$ is not cut-free whereas for Blackburn's system the cut rule is admissible. This is caused by the different rules for the nominals in the two systems. Nevertheless, this does not become a problem since cut elimination already fails for the same reasons as it does with $\mathbf{G}_{\mathcal{H}(@)}$ and the same problems arise when a translation is tried with Blackburn's system.

As for system $\mathbf{G}_{\mathcal{H}(@)}$, all formulae occuring in a proof or a derivation have to be satisfaction statements. Note that in Braüner's original system the rule corresponding to $(\Box R)$ of $\mathbf{G}'_{\mathcal{H}(@)}$ is subject to the condition that j *is a new nominal* in contrast to j *does not occur freely in the conclusion* which applies to the $(\Box R)$-rule. The stronger condition is needed in Braüner's system because his system $\mathcal{H}(@)$ shares the rules with another system for a logic where another hybrid operator is used besides @.

$$\text{(axiom)}\ \frac{}{\vdash @_i\neg j, @_i\neg A, @_j A} \qquad (\top)\ \frac{}{\vdash @_i\top}$$

$$(\wedge R)\ \frac{\vdash \Phi, @_i A \qquad \vdash \Psi, @_i B}{\vdash \Phi, \Psi, @_i(A \wedge B)} \qquad (\vee R)\ \frac{\vdash \Phi, @_i A, @_i B}{\vdash \Phi, @_i(A \vee B)}$$

$$(CR)\ \frac{\vdash \Phi, A, A}{\vdash \Phi, A} \qquad (WR)\ \frac{\vdash \Phi}{\vdash \Phi, A}$$

$$(@R)\ \frac{\vdash \Phi, @_i A}{\vdash \Phi, @_j @_i A} \qquad (\text{ref})^1\ \frac{\vdash \Phi, @_i\neg i}{\vdash \Phi}$$

$$(\Box R)^2\ \frac{\vdash \Phi, @_i\Box\neg j, @_j A}{\vdash \Phi, @_i\Box A} \qquad (\Diamond R)\ \frac{\vdash \Phi, @_i\Box\neg j, @_i\Diamond A, @_j A}{\vdash \Phi, @_i\Box\neg j, @_i\Diamond A}$$

$$(\text{nom1})^3\ \frac{\vdash \Phi, @_j A, @_i\neg j, @_i A}{\vdash \Phi, @_i\neg j, @_i A}$$

$$(\text{quasi-analytic cut})^4\ \frac{\vdash \Phi, @_i A \qquad \vdash \Psi, @_i\neg A}{\vdash \Phi, \Psi}$$

[1] the nominal i occurs in, or below, the conclusion

[2] the nominal j does not occur freely in the conclusion and A is not a nominal

[3] A is an atom (propositional variable or nominal)

[4] the nominal i and the formulae A are subformulae of the end-sequent

Figure 9: The sequent calculus $\mathbf{G}'_{\mathcal{H}(@)}$ for $\mathcal{H}(@)$.

3.3.1 Translation from $\mathbf{G}'_{\mathcal{H}(@)}$ to BH↓↑

The mapping from $\mathbf{G}'_{\mathcal{H}(@)}$ formulae to BH↓↑ structures is defined as for $\mathbf{G}_{\mathcal{H}(@)}$. Theorem 3.1 can be adapted to $\mathbf{G}'_{\mathcal{H}(@)}$ as follows:

Theorem 3.3. For every derivation $\begin{array}{c}\Sigma_1 \quad \dots \quad \Sigma_k \\ \bigtriangledown_{\Delta} \\ \Sigma\end{array}$ in $\mathbf{G}'_{\mathcal{H}(@)}$ there is a derivation $\begin{array}{c}(\underline{\Sigma_1}_S, \dots, \underline{\Sigma_k}_S) \\ {\scriptstyle \Delta'} \Big\| \\ \underline{\Sigma}_S\end{array}$ in $\mathsf{BH}{\downarrow}{\uparrow}$.

Proof. Structural induction on the derivation Δ.

Base cases:

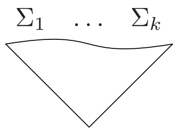
- $\Delta = \Sigma$. Take $\underline{\Sigma}_S$.
- $\Delta = (\top)\,\dfrac{}{\vdash @_i\top}$. Take $e^{@}{\downarrow}\,\dfrac{\mathbf{t}}{@_i\mathbf{t}}$.
- $\Delta = (\text{axiom})\,\dfrac{}{\vdash @_i\neg j, @_i\neg\varphi, @_j\varphi}$. Take

$$\begin{array}{rc}
i{\downarrow} & \mathbf{t} \\ \hline
n^{@}{\downarrow} & \left[@_i\underline{\varphi}_S, @_i\overline{\underline{\varphi}}_S\right] \\ \hline
n{\downarrow} & \left[@_j @_i\underline{\varphi}_S, @_i\overline{\underline{\varphi}}_S\right] \\ \hline
k^{@}{\downarrow} & \left[@_j\left[\bar{i}, \underline{\varphi}_S\right], @_i\overline{\underline{\varphi}}_S\right] \\ \hline
\sigma_n{\downarrow} & \left[@_j\bar{i}, @_j\underline{\varphi}_S, @_i\overline{\underline{\varphi}}_S\right] \\ \hline
 & \left[@_i\bar{j}, @_i\overline{\underline{\varphi}}_S, @_j\underline{\varphi}_S\right]
\end{array}.$$

Inductive cases:

- $\Delta = \begin{array}{c} \begin{array}{cc}\Sigma_1 \ \dots\ \Sigma_k & \Sigma'_1 \ \dots\ \Sigma'_l \\ \bigtriangledown & \bigtriangledown \end{array} \\ (\wedge R)\,\dfrac{\vdash \Phi, @_iA \qquad \vdash \Psi, @_iB}{\vdash \Phi, \Psi, @_i(A\wedge B)}\end{array}$. By induction hypothesis we have two derivations $\begin{array}{c}(\underline{\Sigma_1}_S, \dots, \underline{\Sigma_k}_S) \\ {\scriptstyle \Delta_1} \Big\| {\scriptstyle \mathsf{BH}\downarrow\uparrow} \\ [\underline{\Phi}_S, @_i\underline{A}_S]\end{array}$ and $\begin{array}{c}(\underline{\Sigma'_1}_S, \dots, \underline{\Sigma'_l}_S) \\ {\scriptstyle \Delta_2} \Big\| {\scriptstyle \mathsf{BH}\downarrow\uparrow} \\ [\underline{\Psi}_S, @_i\underline{B}_S]\end{array}$ which are plugged into contexts $\Delta'_1 = (\Delta_1, \underline{\Sigma'_1}_S, \dots, \underline{\Sigma'_l}_S)$ and $\Delta'_2 = ([\underline{\Phi}_S, @_i\underline{A}_S], \Delta_2)$ to obtain the derivation

$$
\begin{array}{rc}
 & (\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S, \underline{\Sigma'_1}_S, \ldots, \underline{\Sigma'_l}_S) \\
 & \Delta'_1 \, \Big\| \, \mathsf{BH}{\downarrow}{\uparrow} \\
 & ([\underline{\Phi}_S, @_i\underline{A}_S], \underline{\Sigma'_1}_S, \ldots, \underline{\Sigma'_l}_S) \\
 & \Delta'_2 \, \Big\| \, \mathsf{BH}{\downarrow}{\uparrow} \\
s & ([\underline{\Phi}_S, @_i\underline{A}_S], [\underline{\Psi}_S, @_i\underline{B}_S]) \\ \hline
s & [\underline{\Psi}_S, ([\underline{\Phi}_S, @_i\underline{A}_S], @_i\underline{B}_S)] \\ \hline
k^{@}{\uparrow} & [\underline{\Phi}_S, \underline{\Psi}_S, (@_i\underline{A}_S, @_i\underline{B}_S)] \\ \hline
 & [\underline{\Phi}_S, \underline{\Psi}_S, @_i(\underline{A}_S, \underline{B}_S)]
\end{array}
$$

- $\Delta =$ 

$$(\vee R)\,\dfrac{\vdash \Phi, @_iA, @_iB}{\vdash \Phi, @_i(A \vee B)}$$

. By induction hypothesis we have a derivation

$$
\begin{array}{c}
(\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S) \\
\Delta \, \Big\| \, \mathsf{BH}{\downarrow}{\uparrow} \\
[\underline{\Phi}_S, @_i\underline{A}_S, @_i\underline{B}_S]
\end{array}
$$

which leads to the derivation

$$
\begin{array}{rc}
 & (\underline{\Sigma_1}_S, \ldots, \underline{\Sigma_k}_S) \\
 & \Delta \, \Big\| \, \mathsf{BH}{\downarrow}{\uparrow} \\
= & [\underline{\Phi}_S, @_i\underline{A}_S, @_i\underline{B}_S] \\ \hline
w{\downarrow} & [\underline{\Phi}_S, @_i[\underline{A}_S, \mathbf{f}], @_i[\underline{B}_S, \mathbf{f}]] \\ \hline
w{\downarrow} & [\underline{\Phi}_S, @_i[\underline{A}_S, \underline{B}_S], @_i[\mathbf{f}, \underline{B}_S]] \\ \hline
c{\downarrow} & [\underline{\Phi}_S, @_i[\underline{A}_S, \underline{B}_S], @_i[\underline{A}_S, \underline{B}_S]] \\ \hline
 & [\underline{\Phi}_S, @_i[\underline{A}_S, \underline{B}_S]]
\end{array}
$$

For the rest of the rules only the relevant derivations are shown.

- (CR) leads to

$$c{\downarrow}\,\dfrac{[\underline{\Phi}_S, \underline{A}_S, \underline{A}_S]}{[\underline{\Phi}_S, \underline{A}_S]}$$

- (WR) leads to

$$
\begin{array}{rc}
= & \underline{\Phi}_S \\ \hline
w{\downarrow} & [\underline{\Phi}_S, \mathbf{f}] \\ \hline
 & [\underline{\Phi}_S, \underline{A}_S]
\end{array}
$$

- $(@R)$ leads to

$$n^{@}{\downarrow}\,\dfrac{[\underline{\Phi}_S, @_i\underline{A}_S]}{[\underline{\Phi}_S, @_j@_i\underline{A}_S]}$$

- (ref) leads to

$$
\begin{array}{rc}
r{\uparrow} & [\underline{\Phi}_S, @_i\bar{i}] \\ \hline
= & [\underline{\Phi}_S, \mathbf{f}] \\ \hline
 & \underline{\Phi}_S
\end{array}
$$

- $(\Box R)$ leads to

$$
gv\downarrow\,\dfrac{[\underline{\Phi_S}, @_i\Box\bar{j}, @_j\underline{A_S}]}{[\underline{\Phi_S}, @_i\Box\underline{A_S}]}
$$

- $(\Diamond R)$ leads to

$$
c\downarrow\,\dfrac{k^{@}\downarrow\,\dfrac{k^{\Box}\downarrow\,\dfrac{n\downarrow\,\dfrac{n^{\Box}\downarrow\,\dfrac{n^{@}\downarrow\,\dfrac{[\underline{\Phi_S}, @_j\underline{A_S}, @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}{[\underline{\Phi_S}, @_i@_j\underline{A_S}, @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}}{[\underline{\Phi_S}, @_i\Box @_j\underline{A_S}, @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}}{[\underline{\Phi_S}, @_i\Box[\bar{j}, \underline{A_S}], @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}}{[\underline{\Phi_S}, @_i[\Box\bar{j}, \Diamond\underline{A_S}], @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}}{[\underline{\Phi_S}, @_i\Box\bar{j}, @_i\Diamond\underline{A_S}, @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}}{[\underline{\Phi_S}, @_i\Box\bar{j}, @_i\Diamond\underline{A_S}]}
$$

- $(nom1)$ leads to

$$
c\downarrow\,\dfrac{k^{@}\downarrow\,\dfrac{n\downarrow\,\dfrac{n^{@}\downarrow\,\dfrac{[\underline{\Phi_S}, @_i\bar{j}, @_i\underline{A_S}, @_j\underline{A_S}]}{[\underline{\Phi_S}, @_i\bar{j}, @_i\underline{A_S}, @_i@_j\underline{A_S}]}}{[\underline{\Phi_S}, @_i\bar{j}, @_i\underline{A_S}, @_i[\bar{j}, \underline{A_S}]]}}{[\underline{\Phi_S}, @_i\bar{j}, @_i\underline{A_S}, @_i\bar{j}, @_i\underline{A_S}]}}{[\underline{\Phi_S}, @_i\bar{j}, @_i\underline{A_S}]}
$$

- (cut) leads to

$$
i\uparrow\,\dfrac{s\,\dfrac{s\,\dfrac{([\underline{\Phi_S}, @_i\underline{A_S}], [\underline{\Psi_S}, @_i\underline{\bar{A}_S}])}{[\underline{\Psi_S}, ([\underline{\Phi_S}, @_i\underline{A_S}], @_i\underline{\bar{A}_S})]}}{[\underline{\Phi_S}, \underline{\Psi_S}, (@_i\underline{A_S}, @_i\underline{\bar{A}_S})]}}{[\underline{\Phi_S}, \underline{\Psi_S}]}
$$

□

Similar to the situation with the translation between $\mathbf{G}_{\mathcal{H}(@)}$ and $\mathsf{BH}\downarrow\uparrow$, cut elimination fails because various up-rules are introduced by the translation. Again only a subset of the rules from $\mathsf{BH}\downarrow\uparrow$ are needed when translating from $\mathbf{G}'_{\mathcal{H}(@)}$ to $\mathsf{BH}\downarrow\uparrow$, namely the rules $i\downarrow$, s, $w\downarrow$, $c\downarrow$, $e^{@}\downarrow$, $n^{@}\downarrow$, $n\downarrow$, $k^{@}\downarrow$, $n^{\Box}\downarrow$, $k^{\Box}\downarrow$, $\sigma_n\downarrow$, $gv\downarrow$, $k^{@}\uparrow$, $r\uparrow$, and $i\uparrow$. Together with Lemma 3.1 we obtain:

Lemma 3.4. The system $\{i\downarrow, i\uparrow, s, w\downarrow, c\downarrow, e^{@}\downarrow, n^{@}\downarrow, n\downarrow, k^{@}\downarrow, n^{\Box}\downarrow, k^{\Box}\downarrow, \sigma_n\downarrow, r\downarrow, gv\downarrow\}$ is complete for $\mathcal{H}(@)$ and the rules $e^{\Box}\downarrow$, $e^{\Box}\uparrow$, $gb\downarrow$, and $gb\uparrow$ are admissible for this system.

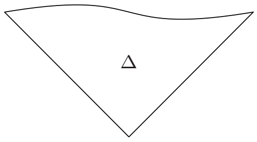
If a Gentzen-Schütte system derived from Blackburn's sequent calculus is used for the translation, one obtains a similar result, which shows that $e^{\Box}\downarrow$ and $e^{\Box}\uparrow$ can be dropped without losing completeness.

3.3.2 Translation from $\mathsf{BH}{\downarrow}{\uparrow}$ to $\mathbf{G}'_{\mathcal{H}(@)}$

The mapping between structures and formulae is defined in the same way as in the corresponding section for system $\mathbf{G}_{\mathcal{H}(@)}$. The Lemma 3.3 can easily be adapted to system $\mathbf{G}'_{\mathcal{H}(@)}$:

Lemma 3.5. For every two formulae A, B and every formula context $C\{\}$ there exists a derivation

$\vdash @_j A, @_j \neg B$



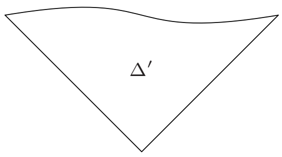
$\vdash @_i C\{A\}, @_i \neg C\{B\}$

in $\mathbf{G}'_{\mathcal{H}(@)}$.

Proof. By structural induction on the context $C\{\}$. The base case for $C\{\} = \{\}$ is trivial. Inductive cases:

- For $C\{\} = \Box C_1\{\}$ the derivation is

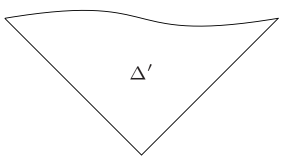
$\Delta =$



$$
(\Box R)\,\dfrac{(\Diamond R)\,\dfrac{(WR)\,\dfrac{(WR)\,\dfrac{\vdash @_j C_1\{A\}, @_j \neg C_1\{B\}}{\vdash @_j C_1\{A\}, @_i \Diamond\neg C_1\{B\}, @_j \neg C_1\{B\}}}{\vdash @_i \Box\neg j, @_j C_1\{A\}, @_i \Diamond\neg C_1\{B\}, @_j \neg C_1\{B\}}}{\vdash @_i \Box\neg j, @_j C_1\{A\}, @_i \Diamond\neg C_1\{B\}}}{\vdash @_i \Box C_1\{A\}, @_i \Diamond\neg C_1\{B\}}\ .
$$

- For $C\{\} = \Diamond C_1\{\}$ the derivation is

$\Delta =$



$$
(\Box R)\,\dfrac{(\Diamond R)\,\dfrac{(WR)\,\dfrac{(WR)\,\dfrac{\vdash @_j C_1\{A\}, @_j \neg C_1\{B\}}{\vdash @_j C_1\{A\}, @_i \Box\neg j, @_j \neg C_1\{B\}}}{\vdash @_j C_1\{A\}, @_i \Diamond C_1\{A\}, @_i \Box\neg j, @_j \neg C_1\{B\}}}{\vdash @_i \Diamond C_1\{A\}, @_i \Box\neg j, @_j \neg C_1\{B\}}}{\vdash @_i \Diamond C_1\{A\}, @_i \Box\neg C_1\{B\}}\ .
$$

- For $C\{\} = @_j C_1\{\}$ the derivation is

$$\Delta = \dfrac{\dfrac{\dfrac{\Delta'}{\vdash @_j C_1\{A\}, @_j \neg C_1\{B\}}}{\vdash @_j C_1\{A\}, @_i @_j \neg C_1\{B\}}\,(@R)}{\vdash @_i @_j C_1\{A\}, @_i @_j \neg C_1\{B\}}\,(@R) .$$

- For $C\{\} = (C_1 \wedge C_2\{\})$ the derivation is

$$\Delta = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash @_i C_1, @_i \neg C_1, @_i \neg i}\,(axiom)}{\vdash @_i C_1, @_i \neg C_1}\,(ref) \qquad \dfrac{\Delta'}{\vdash @_i C_2\{A\}, @_i \neg C_2\{B\}}}{\vdash @_i(C_1 \wedge C_2\{A\}), @_i \neg C_1, @_i \neg C_2\{B\})}\,(\wedge R)}{\vdash @_i(C_1 \wedge C_2\{A\}), @_i(\neg C_1 \vee \neg C_2\{B\})}\,(\vee R) .$$

- For $C\{\} = (C_1 \vee C_2\{\})$ the derivation is

$$\Delta = \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash @_i C_1, @_i \neg C_1, @_i \neg i}\,(axiom)}{\vdash @_i C_1, @_i \neg C_1}\,(ref) \qquad \dfrac{\Delta'}{\vdash @_i C_2\{A\}, @_i \neg C_2\{B\}}}{\vdash @_i C_1, @_i C_2\{A\}, @_i(\neg C_1 \wedge \neg C_2\{B\})}\,(\wedge R)}{\vdash @_i(C_1 \vee C_2\{A\}), @_i(\neg C_1 \wedge \neg C_2\{B\})}\,(\vee R) .$$

The derivations marked with Δ' exist by induction hypothesis. For applications of the $(\Box R)$-rule, the nominal j can always be chosen in such a way that it does not occur freely in the conclusion by taking a new nominal. □

The translation of derivations in $\mathsf{BH}{\downarrow}{\uparrow}$ to derivations in $\mathbf{G}'_{\mathcal{H}(@)}$ is formalized by the following theorem.

Theorem 3.4. For every derivation $\begin{array}{c} Q \\ \Delta \Big\| \\ P \end{array}$ in $\mathsf{BH}{\downarrow}{\uparrow}\backslash\{gv{\downarrow}, gv{\uparrow}\}$ there is a derivation $\begin{array}{c} \vdash @_j \underline{Q}_G \\ \nabla \Delta' \\ \vdash @_i \underline{P}_G \end{array}$ in $\mathbf{G}'_{\mathcal{H}(@)}$.

Proof. The derivation Δ' in $\mathbf{G}'_{\mathcal{H}(@)}$ is constructed by induction on the length of the derivation in $\mathsf{BH}{\downarrow}{\uparrow}\backslash\{gv{\downarrow}, gv{\uparrow}\}$.

Base Case
If Δ is the trivial derivation consisting of a single structure P, i.e. P and Q coincide, then the corresponding derivation is $\vdash @_i \underline{P}_G$ for some nominal i.

Inductive Case
For the inductive case the topmost rule instance in Δ is singled out:

$$\begin{array}{c} Q \\ \Delta \Big\| \; \mathsf{BH}{\downarrow}{\uparrow}\backslash\{gv{\downarrow}, gv{\uparrow}\} \\ P \end{array} = \begin{array}{c} \rho \dfrac{S\{T\}}{S\{R\}} \\ \Delta_0 \Big\| \; \mathsf{BH}{\downarrow}{\uparrow}\backslash\{gv{\downarrow}, gv{\uparrow}\} \\ P \end{array}$$

Now the corresponding derivation in $\mathbf{G}'_{\mathcal{H}(@)}$ is constructed as follows:

$$\begin{array}{c} \nabla \Pi \\ \vdash @_k R, @_k \neg T \\ \nabla \Delta_1 \\ (cut)\dfrac{\vdash @_j S\{R\}, @_j \neg S\{T\} \quad \vdash @_j S\{T\}}{\vdash @_j S\{R\}} \\ \nabla \Delta_2 \\ \vdash @_i P \end{array},$$

where Δ_1 exists by Lemma 3.5 and Δ_2 by induction hypothesis. The proof Π has to be shown for each rule in $\mathsf{BH}{\downarrow}{\uparrow}\backslash\{gv{\downarrow}, gv{\uparrow}\}$ individually:

- $i{\downarrow}$:

$$(WR)\,\dfrac{(\vee R)\,\dfrac{(\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A}}{\vdash @_i(A \vee \neg A)}}{\vdash @_i(A \vee \neg A), @_i\bot}$$

- s:

$$(\vee R)\,\dfrac{(\vee R)\,\dfrac{(\wedge R)\,\dfrac{(\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A} \qquad \Pi'}{\vdash @_i(A \wedge B), @_iC, @_i\neg A, @_i(\neg B \wedge \neg C)}}{\vdash @_i(A \wedge B), @_iC, @_i(\neg A \vee (\neg B \wedge \neg C))}}{\vdash @_i((A \wedge B) \vee C), @_i(\neg A \vee (\neg B \wedge \neg C))}$$

where Π' is the proof

$$(\wedge R)\,\dfrac{(\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iB, @_i\neg B, @_i\neg i}}{\vdash @_iB, @_i\neg B} \qquad (\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iC, @_i\neg C, @_i\neg i}}{\vdash @_iC, @_i\neg C}}{\vdash @_iB, @_iC, @_i(\neg B \wedge \neg C)}$$

- $w{\downarrow}$:

$$(WR)\,\dfrac{(\top)\,\dfrac{}{\vdash @_i\top}}{\vdash @_iA, @_i\top}$$

- $c{\downarrow}$:

$$(CR)\,\dfrac{(\wedge R)\,\dfrac{(\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A} \qquad (\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A}}{\vdash @_iA, @_iA, @_i(\neg A \wedge \neg A)}}{\vdash @_iA, @_i(\neg A \wedge \neg A)}$$

- $e^{\Box}{\downarrow}$:

$$(\Box R)\,\dfrac{(WR)\,\dfrac{(WR)\,\dfrac{(\top)\,\dfrac{}{\vdash @_j\top}}{\vdash @_i\Box\neg j, @_j\top}}{\vdash @_i\Box\neg j, @_j\top, @_i\bot}}{\vdash @_i\Box\top, @_i\bot}$$

- $e^{@}{\downarrow}$:

$$(WR)\,\dfrac{(@R)\,\dfrac{(\top)\,\dfrac{}{\vdash @_i\top}}{\vdash @_j@_i\top}}{\vdash @_j@_i\top, @_j\bot}$$

- $k^{@}{\downarrow}$:

$$(\vee R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(\wedge R)\,\dfrac{(\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A} \qquad (\mathit{ref})\,\dfrac{(\mathit{axiom})\,\dfrac{}{\vdash @_iB, @_i\neg B, @_i\neg i}}{\vdash @_iB, @_i\neg B}}{\vdash @_iA, @_iB, @_i(\neg A \wedge \neg B)}}{\vdash @_iA, @_iB, @_j@_i(\neg A \wedge \neg B)}}{\vdash @_iA, @_j@_iB, @_j@_i(\neg A \wedge \neg B)}}{\vdash @_j@_iA, @_j@_iB, @_j@_i(\neg A \wedge \neg B)}}{\vdash @_j(@_iA \vee @_iB), @_j@_i(\neg A \wedge \neg B)}$$

- $n{\downarrow}$:

$$(\vee R)\,\dfrac{(@R)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_j\neg i, @_jA, @_i\neg A}}{\vdash @_j\neg i, @_jA, @_j@_i\neg A}}{\vdash @_j(\neg i \vee A), @_j@_i\neg A}$$

- $n^{\Box}{\downarrow}$:

$$(\Box R)\,\dfrac{(WR)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A}}{\vdash @_iA, @_j@_i\neg A}}{\vdash @_k@_iA, @_j@_i\neg A}}{\vdash @_j\Box\neg k, @_k@_iA, @_j@_i\neg A}}{\vdash @_j\Box@_iA, @_j@_i\neg A}$$

The nominal k can be chosen in such a way that it does not occur in A.

- $n^{@}{\downarrow}$:

$$(@R)\,\dfrac{(@R)\,\dfrac{(@R)\,\dfrac{(ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_iA, @_i\neg A, @_i\neg i}}{\vdash @_iA, @_i\neg A}}{\vdash @_iA, @_k@_i\neg A}}{\vdash @_j@_iA, @_k@_i\neg A}}{\vdash @_k@_j@_iA, @_k@_i\neg A}$$

- $r{\downarrow}$:

$$(WR)\,\dfrac{(@R)\,\dfrac{(ref)\,\dfrac{(ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_ii, @_i\neg i, @_i\neg i}}{\vdash @_ii, @_i\neg i}}{\vdash @_ii}}{\vdash @_j@_ii}}{\vdash @_j@_ii, @_j\bot}$$

- $\sigma_n{\downarrow}$:

$$(@R)\,\dfrac{(@R)\,\dfrac{(ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_i\neg j, @_ji, @_i\neg i}}{\vdash @_i\neg j, @_ji}}{\vdash @_j@_i\neg j, @_ji}}{\vdash @_j@_i\neg j, @_j@_ji}$$

- $k^{\Box}{\downarrow}$:

$$(\vee R)\,\dfrac{(\Box R)\,\dfrac{(\Diamond R)\,\dfrac{(\wedge R)\,\dfrac{(\Diamond R)\,\dfrac{(WR)\,\dfrac{(WR)\,\dfrac{(WR)\,\dfrac{(ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_jA, @_j\neg A, @_j\neg j}}{\vdash @_jA, @_j\neg A}}{\vdash @_i\Box\neg j, @_jA, @_j\neg A}}{\vdash @_i\Diamond\neg A, @_i\Box\neg j, @_jA, @_j\neg A}}{\vdash @_i\Diamond(A\wedge B), @_i\Diamond\neg A, @_i\Box\neg j, @_jA, @_j\neg A}}{\vdash @_i\Diamond(A\wedge B), @_i\Diamond\neg A, @_i\Box\neg j, @_jA} \quad (ref)\,\dfrac{(axiom)\,\dfrac{}{\vdash @_j\neg B, @_jB, @_j\neg j}}{\vdash @_j\neg B, @_jB}}{\vdash @_i\Diamond(A\wedge B), @_i\Diamond\neg A, @_i\Box\neg j, @_j\neg B, @_j(A\wedge B)}}{\vdash @_i\Diamond(A\wedge B), @_i\Diamond\neg A, @_i\Box\neg j, @_j\neg B}}{\vdash @_i\Diamond(A\wedge B), @_i\Diamond\neg A, @_i\Box\neg B}}{\vdash @_i\Diamond(A\wedge B), @_i(\Diamond\neg A \vee \Box\neg B)}$$

The nominal j can be chosen in such a way that it does not occur in A or B.

- $gb\downarrow$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash @_jA, @_j\neg A, @_j\neg j}(axiom)}{\vdash @_jA, @_j\neg A}(ref)}{\vdash @_jA, @_i\Diamond\neg A, @_j\neg A}(WR)}{\vdash @_i\Box\neg j, @_jA, @_i\Diamond\neg A, @_j\neg A}(WR)}{\vdash @_i\Box\neg j, @_jA, @_i\Diamond\neg A}(\Diamond R)}{\vdash @_i\Box\neg j, @_jA, @_k@_i\Diamond\neg A}(@R)}{\vdash @_i\Box\neg j, @_k@_jA, @_k@_i\Diamond\neg A}(@R)}{\vdash @_k@_i\Box\neg j, @_k@_jA, @_k@_i\Diamond\neg A}(@R)}{\vdash @_k(@_i\Box\neg j \vee @_jA), @_k@_i\Diamond\neg A}(\vee R)
$$

The proofs for the up-rules can be found in a similar way. $\Box$

The reason why Theorem 3.4 only holds for $\mathsf{BH}{\downarrow}{\uparrow}\backslash\{gv\downarrow, gv\uparrow\}$ is that $gv\downarrow$ cannot be translated in all possible cases. While constructing the proof Π for $gv\downarrow$ which would be needed to show that Theorem 3.4 also holds for $\mathsf{BH}{\downarrow}{\uparrow}$, the following situation was encountered:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash @_i\Box\neg j, @_i\Diamond j, @_i\neg i}(axiom)}{\vdash @_i\Box\neg j, @_i\Diamond j}(ref)}{\vdash @_i\Box\neg j, @_j@_i\Diamond j}(@R) \quad \dfrac{\dfrac{\dfrac{}{\vdash @_jA, @_j\neg A, @_j\neg j}(axiom)}{\vdash @_jA, @_j\neg A}(ref)}{\vdash @_jA, @_j@_j\neg A}(@R)}{\vdash @_i\Box\neg j, @_jA, @_j(@_i\Diamond j \wedge @_j\neg A)}(\wedge R)}{\vdash @_i\Box A, @_j(@_i\Diamond j \wedge @_j\neg A)}(\Box R)}{\vdash @_j@_i\Box A, @_j(@_i\Diamond j \wedge @_j\neg A)}(@R)
$$

We can assume that the nominal j does not occur freely in A, since the $gv\downarrow$-rule may only be applied in $\mathsf{BH}{\downarrow}{\uparrow}$ if this is the case. By binding the only remaining free occurrence of j with $@_j$, we can ensure that this part of the condition of the ($\Box$)-rule is fulfilled. However, the proof for $gv\downarrow$ is only correct in cases where A is not a nominal. If it is a nominal, then the condition on the ($\Box$)-rule would be violated. One would have to construct a second proof for this particular case. Unfortunately, no such proof was found. Since neither the (*nom*1)-rule nor the (*quasi-analytic cut*)-rule were used in the proofs for the other rules, it is likely that such a proof - if it exists - would require these two rules. For $gv\uparrow$ the situation is the same.

This problem involving the gv-rules represents a second reason why cut elimination via $\mathbf{G}'_{\mathcal{H}(@)}$ was not achieved.

4 Conclusions

The cut elimination technique which uses a translation between a system in the calculus of structures and a cut-free sequent calculus for the same logic was not successful for $\mathsf{BH}{\downarrow}{\uparrow}$ and $\mathcal{H}(@)$, although it was tried with two different sequent systems. It was not possible to show the admissibility of the cut-rule with this approach, since for both systems the translation from the sequent calculus back to the calculus of structures introduces up-rules other than the cut rule $i{\uparrow}$. In this regard, it was not possible to improve the result by Straßburger [Str07] which already states the completeness of system $\mathsf{BH}{\downarrow}{\cup}\{k^{@}{\uparrow}\}$. However, a way to translate proofs from $\mathsf{BH}{\downarrow}{\uparrow}$ into proofs in one of the sequent systems $\mathbf{G}_{\mathcal{H}(@)}$ and $\mathbf{G}'_{\mathcal{H}(@)}$ (with a minor restriction on $gv{\downarrow}$ and $gv{\uparrow}$), as well as in the opposite direction was found. Furthermore, both translations back to the calculus of structures have shown the admissibility of some of the inference rules which results in two complete systems for $\mathcal{H}(@)$ with a smaller number of rules than $\mathsf{BH}{\downarrow}{\cup}\{k^{@}{\uparrow}\}$. As possible ways to show cut elimination for $\mathsf{BH}{\downarrow}{\uparrow}$, there are several options which could be tried next: A proof similar to the syntactic cut elimination procedures given in [Brü04] and [Str03] could lead to the admissibility of $i{\uparrow}$ itself. Likewise, one could also try to show that $k^{@}{\uparrow}$ is not needed for completeness by induction on the structure of $\mathsf{BH}{\downarrow}{\uparrow}$ proofs. Another option would be to follow Blackburn's proof which uses Hintikka sets to show completeness for his tableau system and adapt it to a completeness proof for $\mathsf{BH}{\downarrow}$.

The second drawback of system $\mathsf{BH}{\downarrow}{\uparrow}$ that the rules $v{\downarrow}$ and $v{\uparrow}$ do not represent valid implications (in the sense that the premise should imply the conclusion), still persists with their generalized versions $gv{\downarrow}$ and $gv{\uparrow}$ which allow an arbitrary context. The rules only represent valid implications if the side condition that j does not occur in the conclusion is fulfilled, i.e. the condition is needed such that the implication holds w.r.t. the Kripke semantics. This implies that it is not possible to get rid of the side condition but it might be the case that the rules are not needed for completeness of the inference system (which could be tried using the same approaches as suggested above for the rules $i{\uparrow}$ and $k^{@}{\uparrow}$).

As already suggested by Straßburger in [Str07], it is straightforward to construct a deep inference system which can also deal with hybrid operators other than @ by transforming the rules from the tableau systems or sequent calculae in [Bla00], [Bra08], or [Sel01] into deep inference rules. But it is likely that such a system will contain rules with side conditions which arise from the side conditions of the corresponding rules in the original systems, similar as it is the case for the gv-rules. Therefore, an important task would be to find out which of these conditions are really needed for completeness and which of them are only used in order to achieve some particular property for the system, e.g. a normalisation result as in Braüner's first system. As a comparison between the two sequent systems by Braüner and Blackburn's system suggests, there is some flexibility with the structure of the rules for the nominals. Hence, an investigation on which set of rules for the nominals is more suitable for a translation into the calculus of structures is advisable.

A further problem for future investigation, which is also presented in [Str07], is the question how axioms for different frame classes can be incorporated in the calculus of structures.

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Statement of Academic Honesty

I hereby declare that I have not used any auxiliary means for my thesis work other than what has been cited in my thesis.

Dresden, October 2008

Armin Troy